Course Plan

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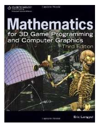
Point and vector algebra
Products: additional reading

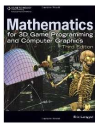
Products between vectors / between versors

- Dot product (or inner product)
  - Output: a scalar



Section 2.2

- Cross product (or vector product)
  - Output: a vector (note: not a versor)



Section 2.3

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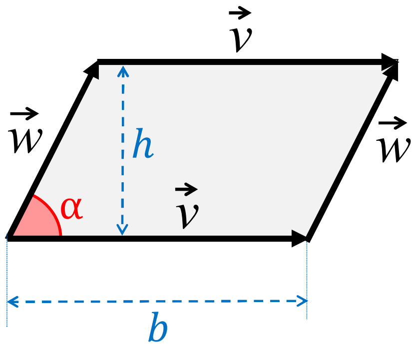
Geometric interpretation: cross product is the parallelogram area



$$\|\vec{v} \times \vec{w}\| = \underbrace{\|\vec{v}\|}_{b} \cdot \underbrace{\|\vec{w}\| \cdot |\sin(\alpha)|}_{h}$$

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Note: Generalization to N - Dimensions

- Everything seen in this lecture generalizes in 2D (for 2D games), or even in N>3 dimensions
- Exception: the cross product is only defined in 3D
  - But in 2D, the problem of finding a vector/versor orthogonal to one (just one!) given vector/versor is easy: *"swap coordinates, flip one* sign"*
    (x,y) orthogonal to (-y,x), and also to (y,-x)

    *: which coordinate you flip determines if you rotate 90° clockwise or counterclockwise: try!

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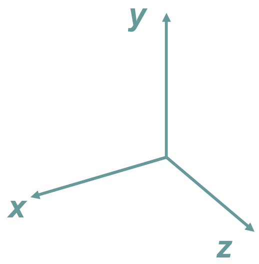
recap: Vector Base



- Axes: set of *n* lin. ind. vectors (***x***,***y***,***z***)
- Any vector ***v*** can be expressed in exactly 1 way as a linar combination of these vectors
- The weights are the coord of ***v*** in that base

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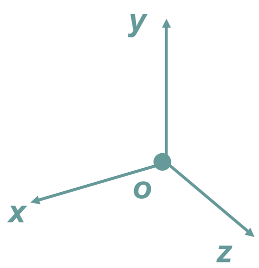
recap: Reference Frame (or Space)



- *n* axes (vectors) (vector base)
  +
  1 origin (point)
- Any vector ***v*** : one linear comb. of the axes
- Any point ***p*** : origin + one linear comb. of axes

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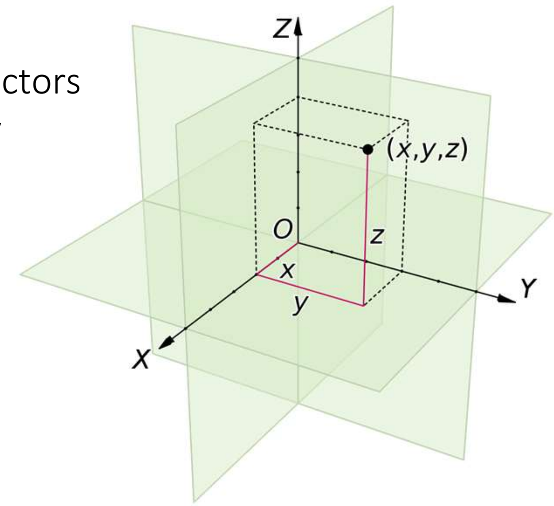
Recap: Orthonormal Frames Or Cartesian Frame

- Axes are unit vectors and reciprocally orthogonal



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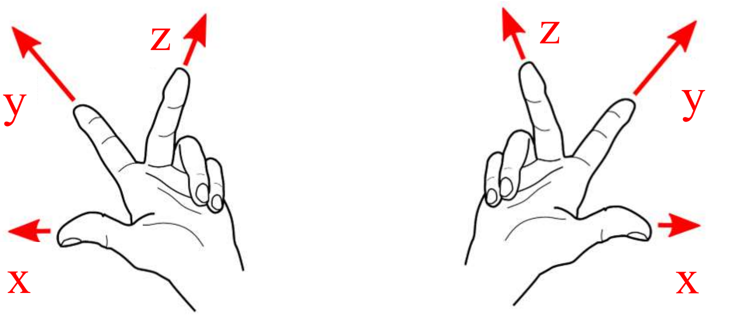
Recap: Handed-ness of a (Cartesian) frame

- They can be right- or left-handed



$x \times y = z$

$x \times y = z$

regardless!

Use the same hand to *imagine* a cross product

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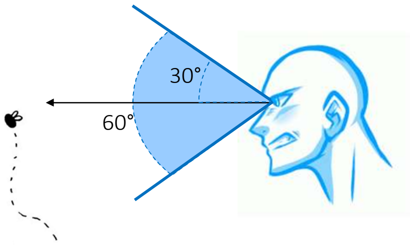
3D videogames

Points, Vectors, Versors: mini task and exercises Part II

Marco Tarini



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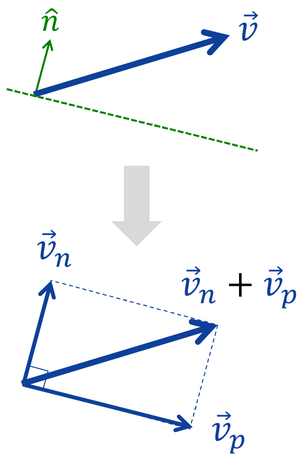
Decompose a vector into components

- Given a vector $\vec{v}$ and a plane normal $\hat{n}$, split $\vec{v}$ in the vector sum $\vec{v} = \vec{v}_n + \vec{v}_p$ with
  - $\vec{v}_n$ orthogonal to the plane (= parallel to $\hat{n}$)
  - $\vec{v}_p$ parallel to the plane (= orthogonal to $\hat{n}$)



- A solution in 3 steps:
  (1) $s_n \leftarrow \vec{v} \cdot \hat{n}$ s_n a (signed) scalar: the extension of $\vec{v}$ along $\hat{n}$
  (2) $\vec{v}_n \leftarrow s_n \, \hat{n}$ component of $\vec{v}$ along $\hat{n}$
  (3) $\vec{v}_p \leftarrow \vec{v} - \vec{v}_n$

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Plane-plane intersection

"Given two 3D planes, find the line they share"

- Input: a point on plane "A" $\mathbf{p}_\mathrm{A}$ and its normal $\hat{\mathrm{n}}_\mathrm{A}$
  a point on plane "B" $\mathbf{p}_\mathrm{B}$ and its normal $\hat{\mathrm{n}}_\mathrm{B}$
- Output:
  a point on the line $\mathbf{q}$ and the line direction $\hat{\mathrm{d}}$

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Ray-sphere intersection

"I shoot a laser from p to direction $\hat{\mathrm{d}}$. Do I hit a sphere in position q of radius r ? Where?"

- Data: p, q points, r scalar, $\hat{\mathrm{d}}$ versor
- Trace:
  - Hit-point is s on laser ray:
    $\mathrm{s} = \mathrm{p} + k\,\hat{\mathrm{d}}$, for some unknown scalar $k \geq 0$
  - Hit-point is s on sphere:
    $\|\mathrm{q}-\mathrm{s}\| = r \quad \leftrightarrow \quad (\mathrm{q}-\mathrm{s}) \cdot (\mathrm{q}-\mathrm{s}) = r^2$
  - Combine the two equations (substitute s in second),
    solve for k (it's a 2^{nd} degree equation),
    test that k exists and that it is >0)

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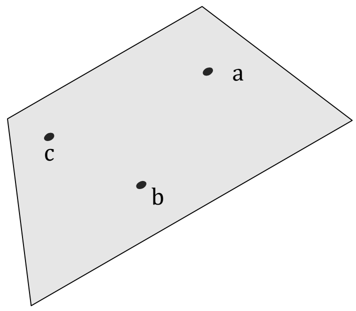
Problem: surface normal (trivial)

"I have three points on $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ on a plane: find the normal $\hat{\mathbf{n}}$ of this plane (a versor)"

- Trace:
  find any two
  different vectors
  on the plane
  …
- Question: what determines the direction of $\hat{\mathbf{n}}$?



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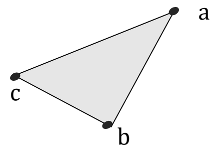
Problem: triangle area (trivial)

"I have three points on $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in space.
Find the area of the triangle connecting them"

- Hint:
  it's half the area
  of a parallelogram



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Vector orthogonalization

"Find a versor $\hat{u}'$ that is ortogonal to a given $\hat{n}$ such that it is as similar as possible to a given versor $\hat{u}$"

Solution: $\hat{u}' = \hat{n} \times \hat{u} \times \hat{n}$, then renormalize it.

Coding examples, in different languages:

GLSL
```
vec3 n,u;
u = cross( cross( n , u ) , n );
u = normalize( u );
```

C++, with UE
```
FVector n,u;
u = FVector::CrossProduct( FVector::CrossProduct(n,u),n );
u.Normalize();
```

C#, with Unity
```
Vector3 n,u;
u = Vector3.Cross( Vector3.Cross( n , u ) , n );
u = u.normalized;
```

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Orthonormal base completion

"I have only two axes $\hat{x}$ and $\hat{y}$ of an orthonormal bases, how do I find the third vector $\hat{z}$?"

- Data: $\hat{x}$, $\hat{y}$ versors
- Hypotheses: $\hat{x}$ and $\hat{y}$ are already orthogonal
- Variant: $\hat{y}$ is not exactly orthogonal to $\hat{x}$, but I want to change it the least to make it orthogonal ($\hat{x}$ is to be kept constant)
  (see previous problem)

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Shooting a walking target (with a finite speed bullet) 1/2

"I shoot a bullet from p with velocity $\vec{\mathrm{v}}$. At which time the bullet will be the closest to a target currently in position q and moving with velocity $\vec{\mathrm{w}}$? Where will bullet and target be, at that point?"

- Data: p, q points, $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$ vectors
- Hypothesis: nothing accelerates (everything keeps moving at a constant speed)

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Shooting a walking target (with a finite speed bullet) 2/2

Trace

- Position of bullet at time t : $\mathrm{p} + t\,\vec{\mathrm{v}}$
- Position of target at time t : $\mathrm{q} + t\,\vec{\mathrm{w}}$
- Squared distance between the two at time t :

$$\| (\mathrm{p} + t\,\vec{\mathrm{v}}) - (\mathrm{q} + t\,\vec{\mathrm{w}}) \|^2$$

$$=$$

$$\| (\mathrm{p} - \mathrm{q}) + t\,(\vec{\mathrm{v}} - \vec{\mathrm{w}}) \|^{2}$$

- Work on formulas (remember that $\|\vec{\mathrm{v}}\|^2 = \vec{\mathrm{v}} \cdot \vec{\mathrm{v}}$)
  find derivative for dt,
  equate derivative to 0, extract t

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