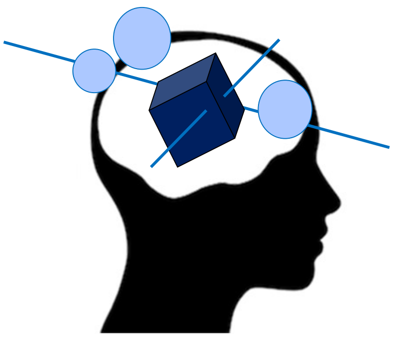
3D videogames

Additional notes on dual-quaternions

(not part of the exam, but maybe useful in life)



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Dual Quaternions: what we said so far 1/2

- New “fantasy” assumption: there is a ε such that $\varepsilon \neq 0$, $\varepsilon^2 = 0$
- A dual quaternion: p + ε q , with p,q ∈ $\mathbb{H}$ ← quaternion set
- So, eight scalars (*a*,*b*,*c*,*d*,*e*,*f*,*g*,*h*)
  - weights for: 1, i , j , k , ε , εi , εj , εk

$$\underbrace{\overbrace{a}^{\text{real part of p}} + \overbrace{b\,i + c\,j + d\,k}^{\text{imaginary part of p}}}_{\text{p (the “primal” quaternion)}} + \underbrace{\overbrace{e\,\varepsilon}^{\text{real part of q}} + \overbrace{f\,\varepsilon\,i + g\,\varepsilon\,j + h\,\varepsilon\,k}^{\text{imaginary part of q}}}_{\varepsilon\text{q (the “dual” quaternion)}}$$

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Dual Quaternions: what we said so far 2/2

$a + b\,i + c\,j + d\,k$ $\quad$ $e + f\,i + g\,j + h\,k$

- A dual quaternion p + ε q can represent:
  - a point / vector in 3D , when p = 1 and Real(q) = *e* = 0 then Im(q) = (*f*,*g*,*h*) = (*x*,*y*,*z*)
  - a roto-translation, when ||p|| = 1 and p·q = 0 (4D dot product) then p encodes the rotational part and q encodes the translational part
  - (nothing, otherwise)
- To roto-translate a point a with roto-trans b just "conjugate" their representations a' ← b ∗ a ∗ $\overline{\mathrm{b}}$

dual-quaternion conjugate: $\overline{\mathrm{p}}$ - ε $\overline{\mathrm{q}}$

dual quaternion multiplication

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Quaternion math: Dot Product (a reminder of a few rules that will be useful)

- Computed considering the quaternions as vectors in 4D
- Let's denote it as $\langle\, \mathbf{p} \,,\mathbf{q}\, \rangle$ to avoid confusion with the standard quaternion product $\mathbf{p}\ \mathbf{q}$
- The dot can also be rewritten as the real part of the product of $\mathbf{p}$ with the conjugate of $\mathbf{q}$, or vice-versa, any order:
$$\langle\, \mathbf{p} \,,\mathbf{q}\, \rangle = Re(\mathbf{p}\ \overline{\mathbf{q}}) = Re(\ \overline{\mathbf{q}}\ \ \mathbf{p}\ )$$
- Dot product of a quaternion with itself:
$$\langle\, \mathbf{p} \,,\mathbf{p}\, \rangle\ = \mathbf{p}\,\overline{\mathbf{p}} = \|\mathbf{p}\|^2$$
- Also: $(\mathbf{p} + \overline{\mathbf{p}}) = 2\,Re(\mathbf{p})$

  $(\mathbf{p} - \overline{\mathbf{p}}) = 2\,Im(\mathbf{p})$
- Also: $\overline{(\mathbf{p}\ \mathbf{q})} = \overline{\mathbf{q}}\ \ \overline{\mathbf{p}}$

Exercise: understand why (look at the formula of the product!)

Exercise: understand why, including why is the imaginary part 0

Exercise: understand why

Exercise: verify!

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Dual Quaternion math: Product

$$(\mathbf{p}_0 + \varepsilon\, \mathbf{q}_0) * (\mathbf{p}_1 + \varepsilon\, \mathbf{q}_1)$$
$$=$$
$$\mathbf{p}_0\, \mathbf{p}_1 + \varepsilon\,(\mathbf{p}_0\, \mathbf{q}_1 + \mathbf{q}_0\, \mathbf{p}_1) + \cancel{\varepsilon^2 \mathbf{q}_0\, \mathbf{q}_1}$$

Naturally, it isn't commutative (or anticommutative), but it's associative.
Notation: we will always denote the dual-quat multiplication with $*$.
(observe that we won't need the dot product (in 8D) for dual-quats)

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Dual Quaternion math: Conjugate

$\mathbf{a} =$

	imaginary			real
primal	a	b	c	d
dual	e	f	g	h

$\bar{\mathbf{a}} =$

	imaginary			real
primal	− a	− b	− c	d
dual	e	f	g	− h

Rationale: conjugate both primal and dual quat, flip sign of dual quat: $\overline{\mathbf{p} + \varepsilon\, \mathbf{q}} = \bar{\mathbf{p}} - \varepsilon\, \bar{\mathbf{q}}$

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Dual Quaternions of pure rotations & translations

- Pure rotation dual quaternion by axis $\hat{a}$, angle α:

	imaginary	real
primal	$\sin\left(\frac{\alpha}{2}\right)\hat{a}$	$\cos\left(\frac{\alpha}{2}\right)$
dual	0	0

- Pure translation dual quaternion by vector $\vec{t}$:

	imaginary	real
primal	0	1
dual	$\frac{1}{2}\vec{t}$	0

Exercise: check that, in both cases...
- the primal has norm 1
- the primal dot the dual is 0

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Proof 1/2

the dual qual representing a pure translation by $\vec{t}$ — the dual qual representing a 3D point of coords $\vec{v}$ — the conjugate of

$$\left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right) * \left(1 + \varepsilon\,(\vec{v},0)\right) * \left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right) =$$

$$= \left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right) * \left(1 + \varepsilon\left(\vec{v} + \frac{1}{2}\vec{t},0+0\right)\right) =$$

$$= 1 + \varepsilon\left(\vec{v} + \frac{2}{2}\vec{t},0\right)$$

dual-quat representing 3D point $\vec{v} + \vec{t}$

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Proof 2/2

$$\underbrace{(\mathbf{r} + \varepsilon\, 0)}_{\text{dual-quat representing a pure rotation}} * \underbrace{\left(1 + \varepsilon\, (\vec{\mathrm{v}}, 0)\right)}_{\text{dual-quat representing 3D point } \vec{\mathrm{v}}} * \underbrace{(\bar{\mathbf{r}} + \varepsilon\, 0)}_{\text{conjugate of}} =$$

$$= (\mathbf{r} + \varepsilon\, 0) * (\bar{\mathbf{r}} + \varepsilon\, (\vec{\mathrm{v}}, 0)\, \bar{\mathbf{r}}) =$$

$$= \underbrace{\cancel{\mathbf{r}\,\bar{\mathbf{r}}}^{\,1} + \varepsilon \underbrace{(\mathbf{r}\, (\vec{\mathrm{v}}, 0)\, \bar{\mathbf{r}})}_{\text{quat representing the rotation } \vec{\mathrm{v}}' \text{ of } \vec{\mathrm{v}}}}_{\text{Dual-quat representing } \vec{\mathrm{v}}'}$$

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Dual Quaternion for a roto-translation

$$\underbrace{(1 + \varepsilon\, \mathbf{t})}_{\text{then, translate}} * \underbrace{(\mathbf{r} + \varepsilon\, 0)}_{\text{first, rotate}}$$

$$=$$

$$\mathbf{r} + \varepsilon\, \mathbf{t}\, \mathbf{r}$$

		imaginary	real
with…	$\mathbf{t}$ =	$\frac{1}{2}\vec{\mathrm{t}}$	0
	$\mathbf{r}$ =	$s\, \hat{\mathrm{a}}$	c
and so…	$\mathbf{t}\,\mathbf{r} = \frac{1}{2}$	$s\ \vec{\mathrm{t}} \times \hat{\mathrm{a}}\ +\ c\, \vec{\mathrm{t}}$	$-s\, \hat{\mathrm{a}} \cdot \vec{\mathrm{t}}$

$s = \sin\left(\frac{\alpha}{2}\right)$

$c = \cos\left(\frac{\alpha}{2}\right)$

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Conclusion: the dual quat for the rotation around $\hat{a}$ by α followed by trans by $\vec{t}$

	imaginary	*real*
primal	$s\ \hat{a}$	c
dual	$s\ \vec{v} \times \hat{a}\ +\ c\ \vec{v}$	$-s\ \hat{a} \cdot \vec{v}$

$$s = \sin\left(\frac{1}{2}\alpha\right)$$

$$c = \cos\left(\frac{1}{2}\alpha\right)$$

$$\vec{v} = \frac{1}{2}\vec{t}$$

Exercise: check that…

- the primal has norm 1
- the primal *dot* the dual is 0

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Dual quat applied to vectors 1/2

(they DON'T translate!)

$$\underbrace{\left(1 + \varepsilon\left(\frac{1}{2}\vec{t}, 0\right)\right)}_{\text{the dual qual representing a pure translation by } \vec{t}} * \underbrace{\left(0 + \varepsilon\left(\vec{v}, 0\right)\right)}_{\text{the dual qual representing a 3D vector of coords } \vec{v}} * \underbrace{\left(1 + \varepsilon\left(\frac{1}{2}\vec{t}, 0\right)\right)}_{\text{the conjugate of}} =$$

$$= \left(1 + \varepsilon\left(\frac{1}{2}\vec{t}, 0\right)\right) * \left(0 + \varepsilon\left(\vec{v}, 0\right)\right) =$$

$$= \underbrace{0 + \varepsilon\left(\vec{v}, 0\right)}_{\text{dual-quat representing 3D point } \vec{v}}$$

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dual-quat representing a pure rotation

dual-quat representing specifically a vector $\vec{v}$

conjugate of

$$(\mathbf{r} + \varepsilon\, 0) * \left(0 + \varepsilon\, (\vec{v}, 0)\right) * (\bar{\mathbf{r}} + \varepsilon\, 0) =$$

$$= (\mathbf{r} + \varepsilon\, 0) * (0 + \varepsilon\, (\vec{v}, 0)\, \bar{\mathbf{r}}) =$$

$$= 0 + \varepsilon(\mathbf{r}\, (\vec{v}, 0)\, \bar{\mathbf{r}})$$

quat representing the rotation $\vec{v}'$ of $\vec{v}$

Dual-quat representing $\vec{v}'$

Dual quat applied to vectors 2/2

(they rotate!)

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Inverting a Dual Quaternion roto-translation: conjugating both primal and dual quaterions

a dual quat representing a roto-translation

the dual-quat representing the inverse

Proof:

$$(\mathbf{r} + \varepsilon\, \mathbf{t}) * (\bar{\mathbf{r}} + \varepsilon\, \bar{\mathbf{t}})$$

$$=$$

$$\mathbf{r}\,\bar{\mathbf{r}} + \varepsilon\,(\mathbf{t}\,\bar{\mathbf{r}} + \mathbf{r}\,\bar{\mathbf{t}})$$

1 because $\mathbf{r}$ is unitary

$\mathbf{t}\,\bar{\mathbf{r}}$ plus its conjugate (why?) that is, twice the real part of $\mathbf{t}\,\bar{\mathbf{r}}$ (why?) that is, twice $< \mathbf{t}, \mathbf{r} >$ that is, 0 because it's a roto-translation

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Dual quaternions as rototranslation (summary of other operations)

- Each roto-translation is expressed by exactly two (opposite!) dual quaternions
- Their primal is unitary, their dual is orthogonal to primal
- Dual quaternion 1 (that is, $1 + \varepsilon\, 0$) is the identity (as so is -1)
- Cumulation: multiplication (second $*$ first)
- Inversion: conjugation of both primal and dual (not the same as: dual-quat conjugation)

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Extracting the translation of a Dual Quaternion

Let's just see where the origin goes:

$$\underbrace{(\mathbf{r} + \varepsilon\, \mathbf{t})}_{\text{a dual quat representing a roto-translation}} * \underbrace{\cancel{1}}_{\text{the dual quat representing the origin}} * \underbrace{(\bar{\mathbf{r}} - \varepsilon\, \bar{\mathbf{t}})}_{\text{the conjugated of}}$$

$$=$$

$$\underbrace{\mathbf{r}\,\bar{\mathbf{r}}}_{\substack{1 \\ \text{because } \mathbf{r} \\ \text{is unitary}}} + \varepsilon\, \underbrace{(\mathbf{t}\,\bar{\mathbf{r}} - \mathbf{r}\,\bar{\mathbf{t}})}_{\substack{\mathbf{t}\,\bar{\mathbf{r}} - \overline{\mathbf{t}\,\bar{\mathbf{r}}} \\ \text{therefore twice the} \\ \text{imaginary part of } \mathbf{t}\,\bar{\mathbf{r}}}}$$

$$=$$

$$1 + \varepsilon\, \underbrace{2Im(\mathbf{t}\,\bar{\mathbf{r}})}_{\text{the 3D vector that we are looking for}}$$

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Dual Quaternion for a .... “translo-rotation” :-D

(compare with roto-translation!)

then, rotate ⟵ first, translate

$$(\mathbf{r} + \varepsilon\, 0) * (1 + \varepsilon\, \mathbf{t})$$
$$=$$
$$\mathbf{r} + \varepsilon\, \mathbf{r}\, \mathbf{t}$$

		imaginary	*real*
with...	$\mathbf{t} =$	$\frac{1}{2}\vec{t}$	0
	$\mathbf{r} =$	$s\,\hat{a}$	c
and so...	$\mathbf{t}\,\mathbf{r} = \frac{1}{2}$	$s\;\hat{a} \times \vec{t} + c\;\vec{t}$	$-s\;\hat{a} \cdot \vec{t}$

$s = \sin\left(\frac{\alpha}{2}\right)$

$c = \cos\left(\frac{\alpha}{2}\right)$

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Interpolating Dual Quaternions

$$\text{mix}(\quad \mathbf{p}_0 + \varepsilon\, \mathbf{q}_0 \quad , \quad \mathbf{p}_1 + \varepsilon\, \mathbf{q}_1 \quad , \quad t \;)$$

1. Take shortest path:
   if $\langle\, \mathbf{p}_0 \,,\, \mathbf{p}_1 \rangle$ negative, then flip *both* $\mathbf{p}_1$ and $\mathbf{q}_1$
2. Interpolate *both* primal & dual (LERP):
   $\mathbf{p} = \text{mix}(\quad \mathbf{p}_0 \;,\; \mathbf{p}_1 \,,\; t \;)$
   $\mathbf{q} = \text{mix}(\quad \mathbf{q}_0 \;,\; \mathbf{q}_1 \,,\; t \;)$
3. Re-enforce $\mathbf{p}$ to be unitary:
   divide *both* $\mathbf{p}$ and $\mathbf{q}$ by $||\mathbf{p}||$
4. Re-enforce $\mathbf{q}$ to be orthogonal to $\mathbf{p}$:
   subtract $\langle\, \mathbf{p} \,,\mathbf{q}\, \rangle\, \mathbf{p}$ from $\mathbf{q}$ (why?)

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