Course Plan

today:
QUATERNIONS! ½

- lec. 1: Introduction 🟢
- lec. 2: **Mathematics** for 3D Games 🟢🟢🟢🟢🟡🔵
- lec. 3: **Scene Graph** 🔵
- lec. 4: Game **3D Physics** 🔵🔵🔵◖ + ◗🔵
- lec. 5: Game **Particle Systems** ◖
- lec. 6: Game **3D Models** ◗🔵
- lec. 7: Game **Textures** 🔵🔵
- lec. 9: Game **Materials** ◖
- lec. 8: Game **3D Animations** ◗🔵🔵
- lec. 10: **Networking** for 3D Games 🔵
- lec. 11: **3D Audio** for 3D Games 🔵
- lec. 12: **Rendering Techniques** for 3D Games 🔵
- lec. 13: **Artificial Intelligence** for 3D Games 🔵

A flashback:
Complex Numbers in a nutshell 1/3

- It all starts with a «fantasy» assumption, which is: there is an imaginary number i such that $i^2 = -1$
  - And for any other purpose, i behaves just like a (non-zero) Real number
- Consequences:
  - We now have number of the form $\overset{\text{real part}}{a} + \overset{\text{imaginary part}}{b}\, i$, with $a, b \in \mathbb{R}$, called complex numbers (the set is $\mathbb{C}$)
  - The algebra of complex numbers (how to sum, multiply, invert them…) is simply determined by the «fantasy» assumption above

A flashback:
Complex Numbers in a nutshell 2/3

- For example, sum:

real part ↘ imaginary part ↙

$$(a + b\,i) + (c + d\,i) = (a + c) + (b + d)i$$

- For example, product (remembering $i^2 = -1$):

$$(a + b\,i) * (c + d\,i) = (ac - bd) + (ad + bc)i$$

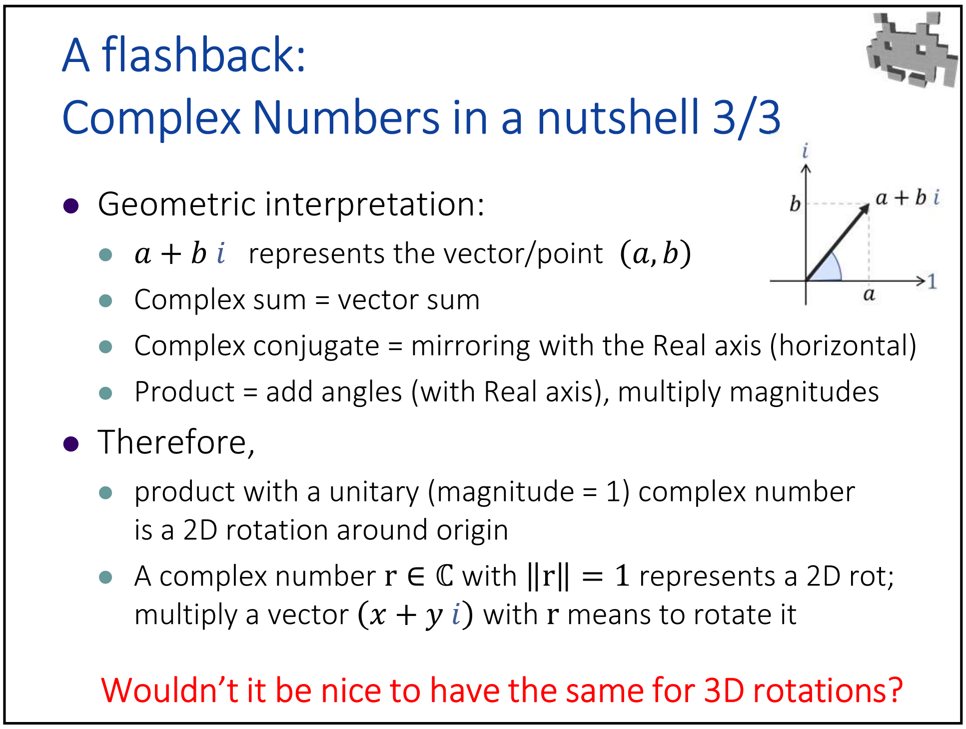
- For example, inverse (check):

$$(a + b\,i)^{-1} = \frac{(a\ - b\,i)}{a^2 + b^2}$$

the «coniugate» of $(a + b\,i\,)$

the squared «magnitude» of $(a + b\,i\,)$

- What is interesting to us is the geometric interpretation of these objects & operations



A flashback:
Complex Numbers in a nutshell 3/3

- Geometric interpretation:
  - $a + b\,i$ represents the vector/point (a, b)
  - Complex sum = vector sum
  - Complex conjugate = mirroring with the Real axis (horizontal)
  - Product = add angles (with Real axis), multiply magnitudes
- Therefore,
  - product with a unitary (magnitude = 1) complex number is a 2D rotation around origin
  - A complex number $\mathrm{r} \in \mathbb{C}$ with $\|\mathrm{r}\| = 1$ represents a 2D rot; multiply a vector $(x + y\,i)$ with r means to rotate it

Wouldn’t it be nice to have the same for 3D rotations?

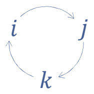
Quaternions

as a table:

×	i	j	k
i	-1	+k	-j
j	-k	-1	+i
k	+j	-i	-1

- New «fantasy» assumption: there are three different "imaginary" numbers i , j , k such that:

$$\begin{cases} i^2 = k^2 = j^2 = -1 \\ ij = k \ , \quad ji = -k \\ jk = i \ , \quad jk = -i \\ ki = j \ , \quad kj = -j \end{cases}$$



  - for any other purpose, i, j, k behave like real numbers
- Consequences:

imaginary parts *real part*

  - We now have number of the form $a\ i + b\ j + c\ k + d$, with $a, b, c, d \in \mathbb{R}$, called Quaternions (their set is $\mathbb{H}$)
  - The algebra of quaternions (how to sum, multiply, invert them…) is simply determined by the «fantasy» assumption
  - Again, what is interesting to us is the geometric interpretation…

Quaternions: how to write them (three equivalent ways)

- Algebraic form: $a\ i + b\ j + c\ k + d$
  - often, omitting the zeros, e.g. $i + 2\ k$ is a quaternion
- As vectors of $\mathbb{R}^4$: $(a\ ,\ b\ ,\ c\ ,\ d\ )$
- As vector & scalar pair: $(\vec{v}\ ,\ d)$

imaginary part, a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *real part, a scalar*

- Conjugate of a quaternion:
  invert the sign of the imaginary parts

Quaternions: algebra

$$q \in \mathbb{H} \qquad\qquad q = ai + bj + ck + d$$

- Sum, Scale, Interpolate , etc.:
  - Trivial

consider them as 4D versors

- Magnitude

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}$$
$$\|q\|^2 = a^2 + b^2 + c^2 + d^2$$

  - «unitary» if it’s 1

Quaternions: algebra

$$q \in \mathbb{H} \qquad\qquad q = ai + bj + ck + d$$

- Product: just apply «fantasy» assumptions
  - Observe: product is not commutative (nor anticommut.)
  - (see next 3 slides for the math)
- «Conjugate»:
  - like for complex numbers: $\bar{q} = -ai - bj - ck + d$

Flip the imaginary parts

- Inverse: (like for complex numbers) $q^{-1} = \bar{q} \,/\, \|q\|^2$
  - For unitary quat, it’s just the conjugate

Quaternions: geometric interpretations

- A quaternion $\mathrm{q} = (\vec{\mathrm{v}}\ ,\ d\ )$ represents :
  - the 3D point / vector / versor $\vec{\mathrm{v}}$, when $d = 0$
  - a 3D rotation, when q is unit, i.e. $\|\mathrm{q}\|^2 = \|\vec{\mathrm{v}}\|^2 + d^2 = 1$
  - neither, otherwise
- If q is a rotation and p is a point ($\mathrm{q}, \mathrm{p} \in \mathbb{H}$) then…
  - $\mathrm{q} \cdot \mathrm{p} \cdot \bar{\mathrm{q}}$ is the rotated point / vector
  - $\bar{\mathrm{q}}$ is the inverse rotation
  - (so, $\bar{\mathrm{q}} \cdot \mathrm{p} \cdot \mathrm{q}$ is point p rotated… in the *other* direction)
  - $\mathrm{q}_0 \cdot \mathrm{q}_1$ is the composited rotation (first q_1 then q_0)

Rotation composition? Quaternion multiplication!

$\mathrm{q}_0 , \mathrm{q}_1 , \mathrm{p} \in \mathbb{H}$
$\mathrm{q}_0 , \mathrm{q}_1$ represent rotations
p represents a point

$$\overbrace{\mathrm{q}_0 \cdot \overbrace{(\mathrm{q}_1 \cdot \mathrm{p} \cdot \bar{\mathrm{q}}_1)}^{\text{p rotated by q1}} \cdot \bar{\mathrm{q}}_0}^{\text{p rotated by q1, rotated by q0}}$$

product is associative
(like for complex numbers)

$$=$$

$$(\mathrm{q}_0 \cdot \mathrm{q}_1) \cdot \mathrm{p} \cdot (\ \bar{\mathrm{q}}_1 \cdot \bar{\mathrm{q}}_0)$$

$\bar{\mathrm{r}} \cdot \bar{\mathrm{s}} = \overline{\mathrm{s} \cdot \mathrm{r}}$
(rules of quaternions)
(remember: product is not commutative)

$$=$$

$$(\mathrm{q}_0 \cdot \mathrm{q}_1) \cdot \mathrm{p} \cdot \overline{(\mathrm{q}_0 \cdot \mathrm{q}_1)}$$

3D Rotations as Quaternions

- quaternion $\mathbf{q}$ representing the 3D rotation of angle α around axis $\hat{a}$:
  - $\mathrm{q} = \left(\sin\left(\frac{\alpha}{2}\right)\hat{a}\,,\ \cos\left(\frac{\alpha}{2}\right)\right)$

  that is
  - $\mathrm{q} = \sin\left(\frac{\alpha}{2}\right)\hat{a}_x i + \sin\left(\frac{\alpha}{2}\right)\hat{a}_y j + \sin\left(\frac{\alpha}{2}\right)\hat{a}_z k + \cos\left(\frac{\alpha}{2}\right)$
- Observe that $\|\mathrm{q}\|^2 = 1$

verify

Exercise: are the following quaternions unitary?

- $\mathbf{q}_0 \ = \ (0,0,-1,0) \ = \ -j$
- $\mathbf{q}_1 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \ 0.5i \ + \ 0.5j \ + 0.5k \ + 0.5$
- $\mathbf{q}_2 = \ (1,1,1,1) = \ i \ + \ j \ + k \ + 1$

Quaternions: exercises

- Which quaternion encodes a turnabout?
  - (ita: «*un dietrofront*»: turning 180° around the up vector)
- Apply that quaternion to rotate a point in (x,y,z)
  - Use plain quaternion algebra, and algebraic notation
- Which quaternion encodes the identity rotation?
  - Is it the only one? If not, which other does?
  - Verify by applying it (or them)
- Which quaternion encodes a turn of 90° to the left?
- Uses your previous *two* answers to find the quat. encoding turn 45° to the left, *by using interpolation*
  - Do you need SLERP in this case? Is NLERP enough? Why?
  - Verify that the solution is correct using the axis-angle formula

Example: turnabout rotation

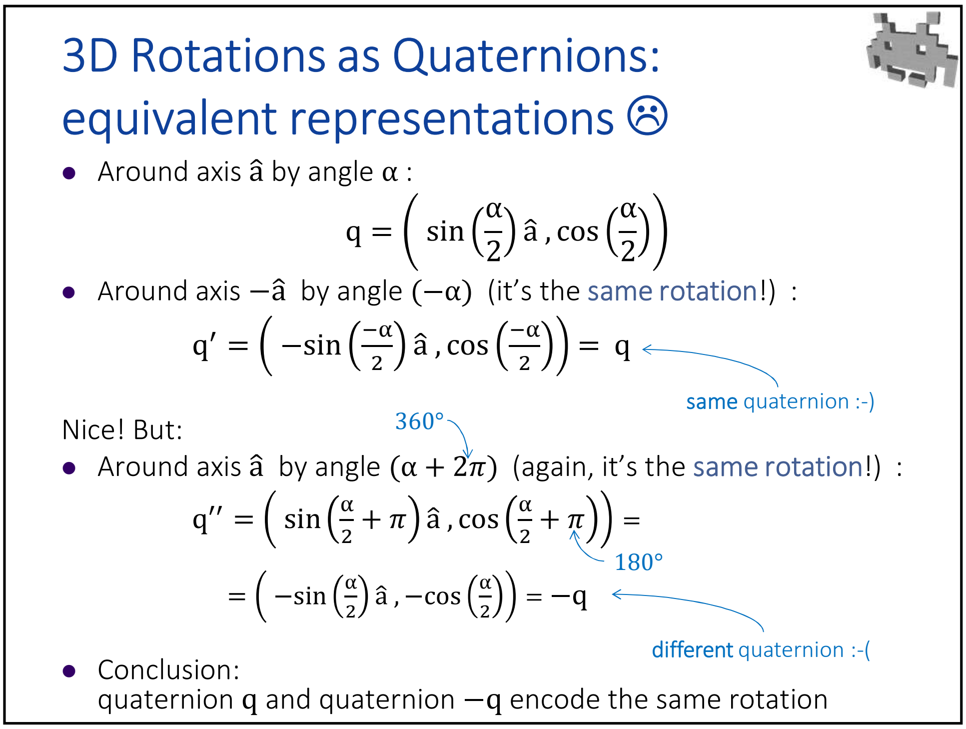
- Find the quaternion $\mathbf{r}$ representing the rotation by 180° (π radiants) around axis Y
  - $\hat{a} = (0,1,0)$
  - $\alpha = \pi, \sin\left(\frac{\alpha}{2}\right) = 1, \cos\left(\frac{\alpha}{2}\right) = 0$
  - $\mathbf{r} = (1\,\hat{a}, 0) = 0i + 1j + 0k + 0 = j$

imaginary vector

real scalar

- Find the quaternion $\mathbf{q}$ representing point $\begin{pmatrix}2\\3\\4\end{pmatrix}$:
  - $\mathbf{q} = 2i + 3j + 4k$
- Rotate that point with that rotation:
  - $\mathbf{q}' = \mathbf{r}\,\mathbf{q}\,\bar{\mathbf{r}} = j\,(2i + 3j + 4k)(-j) = \dots$

(finish me!)



3D Rotations as Quaternions: equivalent representations ☹

- Around axis $\hat{a}$ by angle α :

$$q = \left(\sin\left(\frac{\alpha}{2}\right)\hat{a} \, , \cos\left(\frac{\alpha}{2}\right) \right)$$

- Around axis $-\hat{a}$ by angle $(-\alpha)$ (it's the same rotation!) :

$$q' = \left(-\sin\left(\frac{-\alpha}{2}\right)\hat{a} \, , \cos\left(\frac{-\alpha}{2}\right) \right) = q$$

same quaternion :-)

Nice! But:

360°

- Around axis $\hat{a}$ by angle $(\alpha + 2\pi)$ (again, it's the same rotation!) :

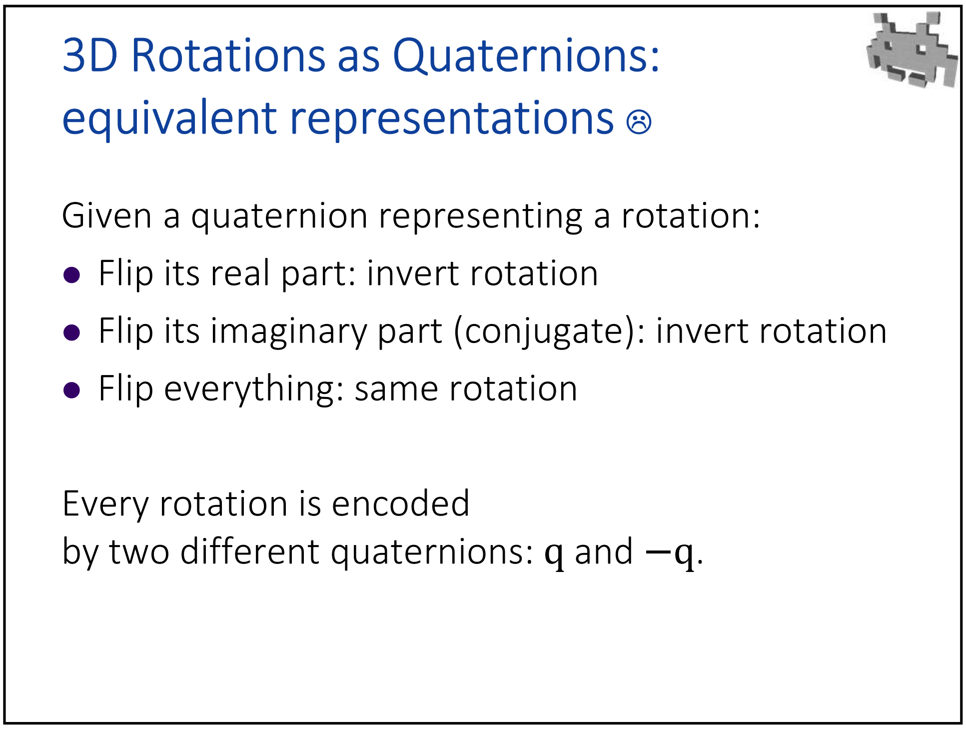
$$q'' = \left(\sin\left(\frac{\alpha}{2} + \pi\right)\hat{a} \, , \cos\left(\frac{\alpha}{2} + \pi\right) \right) =$$

180°

$$= \left(-\sin\left(\frac{\alpha}{2}\right)\hat{a} \, , -\cos\left(\frac{\alpha}{2}\right) \right) = -q$$

different quaternion :-(

- Conclusion:
  quaternion q and quaternion $-q$ encode the same rotation



3D Rotations as Quaternions: equivalent representations ☹

Given a quaternion representing a rotation:

- Flip its real part: invert rotation
- Flip its imaginary part (conjugate): invert rotation
- Flip everything: same rotation

Every rotation is encoded
by two different quaternions: q and $-q$.

Interpolating two quaternions (that represent two rotations)

Good results, but two *caveats*:

⚠ Take the “shortest path” (as usual):
flip 2nd quaternion first, if this makes them closer

- Distance defined as dot product in 4D
(consider quaternions as 4D unit vectors for this)
(remember: dot product between unitary vectors is a measure of similarity!)

⚠ Loss of normality

- Needs re-normalization (NLERP),
- Or SLERP
(again, just consider quaternions as 4D unit vectors)

Shortest path interpolation: the case of quaternions

- Let $\mathbf{p}$ and $\mathbf{q}$ be two rotations
- $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation.
  - Which one to choose?
- Which one is closer to $\mathbf{p}$?
  - Distance between $\mathbf{p}$ and $\mathbf{q}$ = $\text{dot}(\mathbf{p}, \mathbf{q})$
  - Distance between $\mathbf{p}$ and $-\mathbf{q}$ = $\text{dot}(\mathbf{p}, -\mathbf{q})$ = $-\text{dot}(\mathbf{p}, \mathbf{q})$
- Conclusion:
  - If $\text{dot}(\mathbf{p}, \mathbf{q})$ is positive, interpolate with $\mathbf{q}$
  - Otherwise, interpolate with $-\mathbf{q}$

Quaternions, exercise:

Try out rotation cumulation with quaternions

1. Take the quaternion q0 that encodes the 180° rotation around the Y axis (see exercises above)
2. Take the quaternion q1 that encodes the 180° rotation around the Y axis (see exercises above)
3. Compute the quaternion q2 that does the two rotations in succession, in that order
   (using q0 and q1)
4. Which rotation is encoded by q2? Verify with an real 3D object (e.g. a cellphone) that q2 encoded the status that is reached if you rotate by q0 and then q1

Quaternions, exercise:

Try out rotation interpolation with quaternions

1. Take (again) the quaternion q0 that encodes the 180° rotation around the Y axis
2. Take the quaternion q1 that encodes identity rotation (or, rather, *a* quaternion that encodes it)
3. Compute the quaternion q2 that interpolates the two quaternions q0 and q1
   - (what happens with the shortest path?
     why do you think that is?)
4. Which rotation is encoded by q2? To help with the answer, the sin and cos for PI/4 radiants (45°)