3D videogames

Additional notes on dual-quaternions

(not part of the exam, but maybe useful in life)

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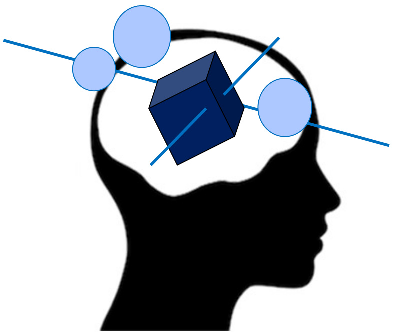

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Dual Quaternions: what we said so far 1/2

- New "fantasy" assumption: there is a ε such that ε ≠ 0 , $\varepsilon^2 = 0$
- A dual quaternion: p + ε q , with p,q ∈ ℍ (quaternion set)
- So, eight scalars (*a*,*b*,*c*,*d*,*e*,*f*,*g*,*h*)
  - weights for: 1, i , j , k , ε , εi , εj , εk

$$\underbrace{\overbrace{a}^{\text{real part of p}} + \overbrace{b\,i + c\,j + d\,k}^{\text{imaginary part of p}}}_{\text{p}} + \underbrace{\overbrace{e\,\varepsilon}^{\text{real part of q}} + \overbrace{f\,\varepsilon\,i + g\,\varepsilon\,j + h\,\varepsilon\,k}^{\text{imaginary part of q}}}_{\varepsilon\,\text{q}}$$

p (the "primal" quaternion) + ε q (the "dual" quaternion)

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Dual Quaternions: what we said so far 2/2

$a + b\,i + c\,j + d\,k$ $\quad$ $e + f\,i + g\,j + h\,k$

- A dual quaternion p + ε q can represent:
  - a point / vector in 3D , when p = 1 and Real(q) = e = 0 then Im(q) = (f,g,h) = (x,y,z)
  - a roto-translation, when ||p|| = 1 and p·q = 0 then p encodes the rotational part and q encodes the translational part
  - (nothing, otherwise)
- To roto-translate a point a with roto-trans b just "conjugate" their representations a' ← b ∗ a ∗ $\overline{\text{b}}$

4D dot product

dual-quaternion conjugate: $\overline{\text{p}}$ - ε $\overline{\text{q}}$

dual quaternion multiplication

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Quaternion math: Dot Product (let's see a few rules that will be useful later)

- It's computed considering the quaternions as vectors in 4D
- Let's denote it as $\langle\, \mathbf{p} \,,\mathbf{q}\, \rangle$ to avoid confusion with the standard quaternion product $\mathbf{p}\ \mathbf{q}$
- The dot can also be rewritten as the real part of the product of $\mathbf{p}$ with the conjugate of $\mathbf{q}$, or vice-versa, any order:
  $$\langle\, \mathbf{p} \,,\mathbf{q}\, \rangle = Re(\mathbf{p}\ \overline{\mathbf{q}}) = Re(\,\overline{\mathbf{q}}\ \ \mathbf{p}\,)$$
- Dot product of a quaternion with itself:
  $$\langle\, \mathbf{p} \,,\mathbf{p}\, \rangle\ = \mathbf{p}\,\overline{\mathbf{p}} = ||\mathbf{p}||^2$$
- Also: $(\mathbf{p} + \overline{\mathbf{p}}) = 2\,Re(\mathbf{p})$
  $(\mathbf{p} - \overline{\mathbf{p}}) = 2\,Im(\mathbf{p})$
- Also: $\overline{(\mathbf{p}\ \mathbf{q})} = \overline{\mathbf{q}}\ \ \overline{\mathbf{p}}$

Exercise: understand why (look at the formula of the product!)

Exercise: understand why, including why is the imaginary part 0

Exercise: understand why

Exercise: verify!

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Dual Quaternion math: Product

$$(\mathbf{p}_0 + \varepsilon\, \mathbf{q}_0) \;*\; (\mathbf{p}_1 + \varepsilon\, \mathbf{q}_1)$$
$$=$$
$$\mathbf{p}_0\, \mathbf{p}_1 + \;\varepsilon\, (\mathbf{p}_0\, \mathbf{q}_1 + \mathbf{q}_0\, \mathbf{p}_1) + \cancel{\varepsilon^2 \mathbf{q}_0\, \mathbf{q}_1}$$

Naturally, it isn't commutative (or anticommutative), but it's associative.
Notation: we will always denote the dual-quat multiplication with $*$.
(observe that we won't need the dot product (in 8D) for dual-quats)

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Dual Quaternion math: Conjugate

$\mathbf{a} =$

	imaginary			real
primal	a	b	c	d
dual	e	f	g	h

$\bar{\mathbf{a}} =$

	imaginary			real
primal	$-a$	$-b$	$-c$	d
dual	e	f	g	$-h$

Rationale: conjugate both primal and dual quat, flip sign of dual quat: $\overline{\mathbf{p} + \varepsilon\,\mathbf{q}} = \bar{\mathbf{p}} - \varepsilon\,\bar{\mathbf{q}}$

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Dual Quaternions of pure rotations & translations

- Pure rotation dual quaternion by axis $\hat{a}$, angle α:

	imaginary	real
primal	$\sin\left(\frac{\alpha}{2}\right)\hat{a}$	$\cos\left(\frac{\alpha}{2}\right)$
dual	0	0

- Pure translation dual quaternion by vector $\vec{t}$:

	imaginary	real
primal	0	1
dual	$\frac{1}{2}\vec{t}$	0

Exercise: check that, in both cases...
- the primal has norm 1
- the primal dot the dual is 0

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Proof 1/2

$$\underbrace{\left(1+\varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}_{\text{the dual qual representing a pure translation by } \vec{t}} * \underbrace{\left(1+\varepsilon\left(\vec{v},0\right)\right)}_{\text{the dual qual representing a 3D point of coords } \vec{v}} * \underbrace{\left(1+\varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}_{\text{the conjugate of}} =$$

$$= \left(1+\varepsilon\left(\frac{1}{2}\vec{t},0\right)\right) * \left(1+\varepsilon\left(\vec{v}+\frac{1}{2}\vec{t},0+0\right)\right) =$$

$$= \underbrace{1+\varepsilon\left(\vec{v}+\frac{2}{2}\vec{t},0\right)}_{\text{dual-quat representing 3D point } \vec{v}+\vec{t}}$$

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Proof 2/2

dual-quat representing a pure rotation; dual-quat representing 3D point $\vec{v}$; conjugate of

$$\underbrace{(\mathbf{r} + \varepsilon\, 0)}_{\text{dual-quat representing a pure rotation}} * \underbrace{\left(1 + \varepsilon\, (\vec{v}, 0)\right)}_{\text{dual-quat representing 3D point } \vec{v}} * \underbrace{(\bar{\mathbf{r}} + \varepsilon\, 0)}_{\text{conjugate of}} =$$

$$= (\mathbf{r} + \varepsilon\, 0) * (\bar{\mathbf{r}} + \varepsilon\, (\vec{v}, 0)\, \bar{\mathbf{r}}) =$$

$$= \underbrace{\mathbf{r}\,\bar{\mathbf{r}}}_{1} + \varepsilon\underbrace{(\mathbf{r}\, (\vec{v}, 0)\, \bar{\mathbf{r}})}_{\text{quat representing the rotation } \vec{v}' \text{ of } \vec{v}}$$

Dual-quat representing rotated 3D point $\vec{v}'$

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Dual Quaternion for a roto-translation

then, translate ← first, rotate

$$\underbrace{(1 + \varepsilon\, \mathbf{t})}_{\text{then, translate}} * \underbrace{(\mathbf{r} + \varepsilon\, 0)}_{\text{first, rotate}}$$

$$=$$

$$\mathbf{r} + \varepsilon\, \mathbf{t}\, \mathbf{r}$$

		imaginary	*real*
with...	$\mathbf{t} =$	$\frac{1}{2}\vec{t}$	0
	$\mathbf{r} =$	$s\, \hat{a}$	c
and so...	$\mathbf{t} * \mathbf{r} = \frac{1}{2}$	$s\ \vec{t} \times \hat{a} + c\, \vec{t}$	$-s\, \hat{a} \cdot \vec{t}$

with...

$s = \sin\left(\frac{\alpha}{2}\right)$

$c = \cos\left(\frac{\alpha}{2}\right)$

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Conclusion: the dual quat for the rotation around $\hat{a}$ by α followed by trans by $\vec{t}$

	imaginary	*real*
primal	$s\,\hat{a}$	c
dual	$s\,\vec{v} \times \hat{a} + c\,\vec{v}$	$-s\,\hat{a} \cdot \vec{v}$

$$s = \sin\left(\frac{1}{2}\alpha\right)$$

$$c = \cos\left(\frac{1}{2}\alpha\right)$$

$$\vec{v} = \frac{1}{2}\vec{t}$$

Exercise: check that...

- the primal has norm 1
- the primal *dot* the dual is 0

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$$\underbrace{\left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}_{\text{the dual qual representing a pure translation by } \vec{t}} * \underbrace{\left(0 + \varepsilon\,(\vec{v},0)\right)}_{\text{the dual qual representing a 3D vector of coords } \vec{v}} * \underbrace{\left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}_{\text{the conjugate of}} =$$

$$= \left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right) * \left(0 + \varepsilon\left(\vec{v},0\right)\right) =$$

$$= \underbrace{0 + \varepsilon\left(\vec{v},0\right)}_{\text{dual-quat representing 3D vector } \vec{v}}$$

Dual quat applied to vectors 1/2

(they DON'T translate!)

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dual-quat representing a pure rotation — dual-quat representing specifically a vector $\vec{v}$ — conjugate of

$$(\mathbf{r} + \varepsilon\, 0) * \left(0 + \varepsilon\, (\vec{v}, 0)\right) * (\bar{\mathbf{r}} + \varepsilon\, 0) =$$

$$= (\mathbf{r} + \varepsilon\, 0) * \left(0 + \varepsilon\, (\vec{v}, 0)\, \bar{\mathbf{r}}\right) =$$

$$= 0 + \varepsilon(\mathbf{r}\, (\vec{v}, 0)\, \bar{\mathbf{r}})$$

quat representing the rotation $\vec{v}'$ of $\vec{v}$

Dual-quat representing $\vec{v}'$

Dual quat applied to vectors 2/2

(they rotate!)

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Inverting a Dual Quaternion roto-translation: conjugating both primal and dual quaterions

a dual quat representing a roto-translation — the dual-quat representing the inverse

Proof:

$$(\mathbf{r} + \varepsilon\, \mathbf{t}) * (\bar{\mathbf{r}} + \varepsilon\, \bar{\mathbf{t}})$$

$$=$$

$$\mathbf{r}\, \bar{\mathbf{r}} + \varepsilon\, (\mathbf{t}\, \bar{\mathbf{r}} + \mathbf{r}\, \bar{\mathbf{t}})$$

1 because $\mathbf{r}$ is unitary

$\mathbf{t}\, \bar{\mathbf{r}}$ plus its conjugate (why?) that is, twice the real part of $\mathbf{t}\, \bar{\mathbf{r}}$ (why?) that is, twice $\langle \mathbf{t}, \mathbf{r} \rangle$ that is, 0 because it's a roto-translation

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Dual quaternions as rototranslation (summary of other operations)

- Each roto-translation is expressed by exactly two (opposite!) dual quaternions
- Their primal is unitary, their dual is orthogonal to primal
- Dual quaternion 1 (that is, 1 + ε 0) is the identity (as so is -1)
- Cumulation: multiplication (second $*$ first)
- Inversion: conjugation of both primal and dual (not the same as: dual-quat conjugation)

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Extracting the translation of a Dual Quaternion

Let's just see where the origin goes:

a dual quat representing a roto-translation; the dual quat representing the origin; the conjugated of

$$(\mathbf{r} + \varepsilon\,\mathbf{t}) * \not{1} * (\bar{\mathbf{r}} - \varepsilon\,\bar{\mathbf{t}})$$

$$=$$

$$\mathbf{r}\,\bar{\mathbf{r}} + \varepsilon\,(\mathbf{t}\,\bar{\mathbf{r}} - \mathbf{r}\,\bar{\mathbf{t}})$$

$\mathbf{r}\,\bar{\mathbf{r}}$: 1 because $\mathbf{r}$ is unitary

$\mathbf{t}\,\bar{\mathbf{r}} - \mathbf{r}\,\bar{\mathbf{t}}$: $\mathbf{t}\,\bar{\mathbf{r}} - \overline{\mathbf{t}\,\bar{\mathbf{r}}}$ therefore twice the imaginary part of $\mathbf{t}\,\bar{\mathbf{r}}$

$$=$$

$$1 + \varepsilon\, 2Im(\mathbf{t}\,\bar{\mathbf{r}})$$

$2Im(\mathbf{t}\,\bar{\mathbf{r}})$: the 3D vector that we are looking for

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Dual Quaternion for a …. "translo-rotation" :-D

(compare with roto-translation!)

then, rotate ⟵ first, translate

$$\underbrace{(\mathbf{r} + \varepsilon\, 0)}_{\text{then, rotate}} * \underbrace{(1 + \varepsilon\, \mathbf{t})}_{\text{first, translate}}$$

$$=$$

$$\mathbf{r} + \varepsilon\, \mathbf{r}\,\mathbf{t}$$

		imaginary	real
with…	$\mathbf{t}$ =	$\frac{1}{2}\vec{t}$	0
	$\mathbf{r}$ =	$s\,\hat{a}$	c
and so…	$\mathbf{t} * \mathbf{r} = \frac{1}{2}$	$s\,\hat{a} \times \vec{t} + c\,\vec{t}$	$-s\,\hat{a} \cdot \vec{t}$

$s = \sin\left(\frac{\alpha}{2}\right)$

$c = \cos\left(\frac{\alpha}{2}\right)$

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Interpolating Dual Quaternions

$$\text{mix}(\ \mathbf{p}_0 + \varepsilon\, \mathbf{q}_0\ ,\ \mathbf{p}_1 + \varepsilon\, \mathbf{q}_1\ ,\ t\ )$$

1. Take shortest path:
   if $\langle\, \mathbf{p}_0\,,\ \mathbf{p}_1 \rangle$ negative, then flip *both* $\mathbf{p}_1$ and $\mathbf{q}_1$
2. Interpolate *both* primal & dual (LERP):
   $\mathbf{p} = \text{mix}(\ \mathbf{p}_0\ ,\ \mathbf{p}_1\ ,\ t\ )$
   $\mathbf{q} = \text{mix}(\ \mathbf{q}_0\ ,\ \mathbf{q}_1\ ,\ t\ )$
3. Re-enforce $\mathbf{p}$ to be unitary:
   divide *both* $\mathbf{p}$ and $\mathbf{q}$ by $\|\mathbf{p}\|$
4. Re-enforce $\mathbf{q}$ to be orthogonal to $\mathbf{p}$:
   subtract $\langle\, \mathbf{p}\,, \mathbf{q} \,\rangle\, \mathbf{p}$ from $\mathbf{q}$ (why?)

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