Course Plan

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Changing the value of dt in Verlet

(whenever it's not constant)

Problem:

if dt now changes to a new dt'

then, all $\mathbf{p}_{old}$ must be updated to some $\mathbf{p}'_{old}$

Find $\mathbf{p}'_{old}$:

$$\vec{v} = (\mathbf{p}_{now} - \mathbf{p}_{old})/dt$$
$$\vec{v} = (\mathbf{p}_{now} - \mathbf{p}'_{old})/dt'$$

current velocity $\vec{v}$ and position $\mathbf{p}_{now}$ must not change

$$\Longrightarrow$$

$$\mathbf{p}'_{old} = \mathbf{p}_{now} \cdot (dt - dt')/dt + \mathbf{p}_{old} \cdot dt'/dt$$

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Velocity damping in Verlet

- Velocity at next frame: $\vec{v} = (\mathbf{p}_{next} - \mathbf{p}_{now})/dt$ (implicit)
- We want to multiply $\vec{v}$ a factor c_{DAMP} (e.g. 0.98 obtained as $(1\text{-}dt\cdot c_{\text{DRAG}})$)
  - before applying accelerations
- We can do that using a more general formula for $\mathbf{p}_{next}$

$$\mathbf{p}_{next} = 2 \cdot \mathbf{p}_{now} - 1 \cdot \mathbf{p}_{old} + dt^2 \cdot \vec{a}$$

$$\mathbf{p}_{next} = (1 + c_{\text{DAMP}}) \cdot \mathbf{p}_{now} - c_{\text{damp}} \cdot \mathbf{p}_{old} + dt^2 \cdot \vec{a}$$

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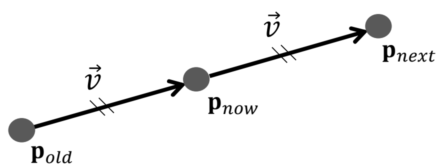
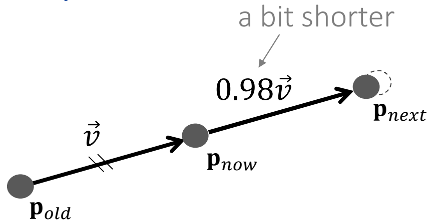
Velocity damping in Verlet (geometric interpretation)



$$\mathbf{p}_{next} = 2 \cdot \mathbf{p}_{now} - 1 \cdot \mathbf{p}_{old}$$

Equivalently,
$\mathbf{p}_{next}$ is an extrapolation of $\mathbf{p}_{now}$, $\mathbf{p}_{old}$:

$$\mathbf{p}_{next} = mix(\mathbf{p}_{old}, \mathbf{p}_{now}, 2)$$



$$\mathbf{p}_{next} = 1.98 \cdot \mathbf{p}_{now} - 0.98 \cdot \mathbf{p}_{old}$$

Equivalently,
$\mathbf{p}_{next}$ is a different extrapolation of $\mathbf{p}_{now}$, $\mathbf{p}_{old}$:

$$\mathbf{p}_{next} = mix(\mathbf{p}_{old}, \mathbf{p}_{now}, 1.98)$$

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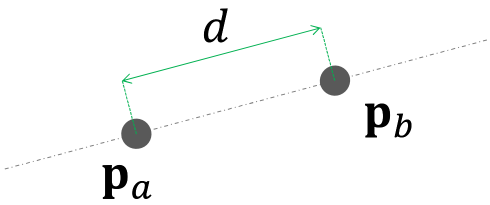
Example of positional constraint: equidistance constraint

*«Particles **a** and **b** must stay at a fixed distance **d** »*



I want that… $\|\mathbf{p}_a - \mathbf{p}_b\| = d$

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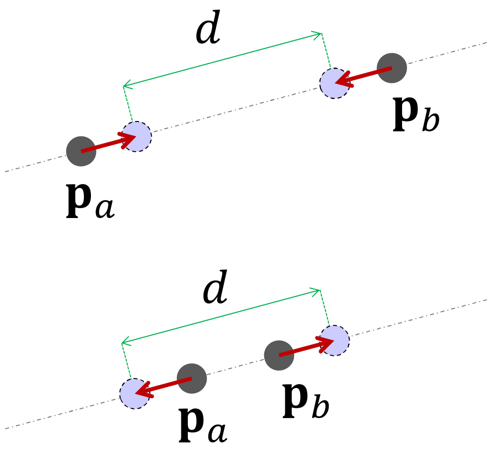
Enforce equidistance constraints
(assuming equal masses for now)

if $\|\mathbf{p}_a - \mathbf{p}_b\| > d$

if $\|\mathbf{p}_a - \mathbf{p}_b\| < d$



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Enforce equidistance constraints: pseudo code

```
Vector3 pa, pb; // curr positions of a,b
float d;        // distance (to enforce)

Vector3 v = pa - pb;
float currDist = v.length;

v /= currDist;  // normalization of v

float delta = currDist - d ;

pa += ( 0.5 * delta) * v;
pb -= ( 0.5 * delta) * v;
```

assuming equal mass, we move each particle *half the way* (see later for the more general case)

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Compare: equidistance constraints vs. springs

- Similar
  - they both mean: these 2 particles "want to be" at *this* distance (not more, not less)
    (some constant scalar parameter *D*)
- but different
  - equidistance constraint:
    - applied during constraint enforcement
    - directly affects positions
    - models a rigid rod between the two particles
      - of a given length
    - sometimes called a "HARD" constraint
  - spring:
    - applied during force evaluation step
    - affects forces, therefore accelerations
    - models a deformable spring between the two particles
      - of a given length
    - sometimes called a "SOFT" constraint
- A physic engine can combine them in one object!

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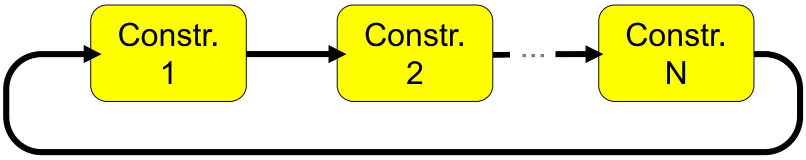
Enforcing sets of constraints

- There are many constraints to impose:
  when you solve one → maybe you break another!
- Simultaneous enforcement: computationally expensive
- Practical & easy solution: just enforce them in cascade
  (similar in concept to Gauss-Seidel solvers):



Repeat until convergence (= max error below threshold)
...but at most for *N* times! (reminder: our simulation is *soft* real-time)

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Enforcing a set of constraints one after the other (in cascade)

- The whole loop for imposing the constraints happen in the constraint enforcement phase on one physics step
- Notes about convergence:
  - needed iterations (typically) few: e.g. 1 ~ 10 (efficient!).
  - if convergence not reached within a given number of steps: never mind, next frames will fix it (it's fairly robust)
  - (it is never reached, if constraints are contradictory)
  - Optimization (to reduce the number of needed iterations): solve the most unsatisfied constraints first
- ⚠ Problem: it's a sequential approach! ☹
  - parallelized versions (similar to Jacobi solvers) are possible
  - they have a worse convergence in practice
    (they require more iterations), but each iteration is faster

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Compounds of particles disguised as rigid bodies

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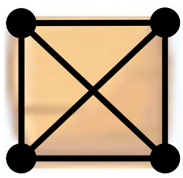
We can combine equidistance constraints to obtain rigid objects!

- Rigid body dynamics as emerging behavior
  - without explicitly keeping track their orientation, angular vel, angular acc., etc.



A box in 2D?
(rigid object)
A configuration of:
- 4 particles
- 6 equidistance constraints

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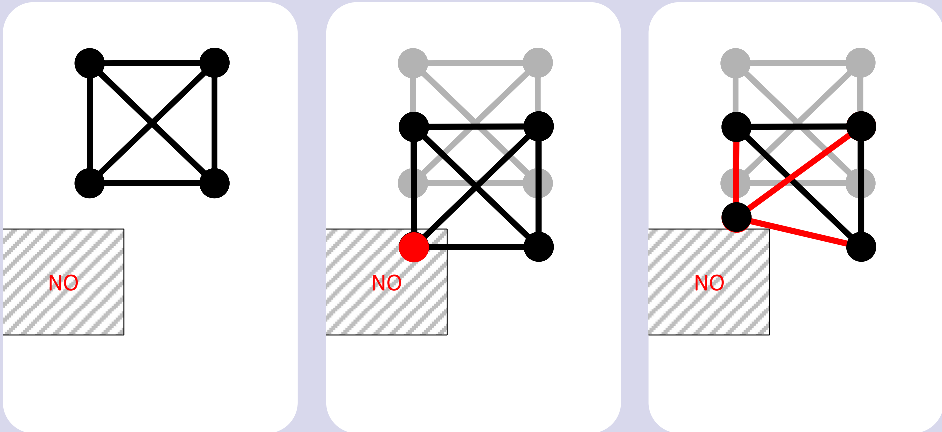
Example



FRAME 0

FRAME 1
before constraints

FRAME 1
after 1st constraint

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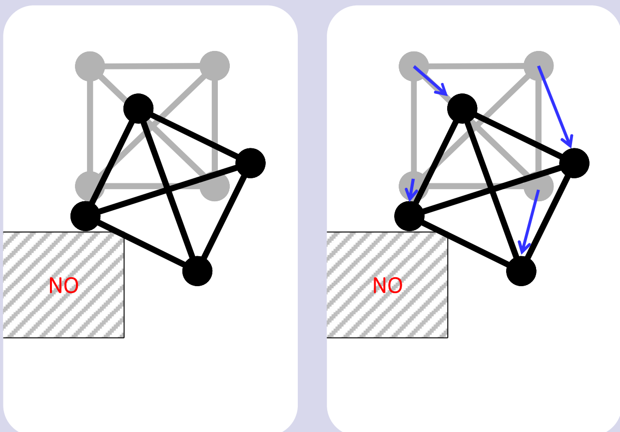
Example



FRAME 1
after all constraints
multiple times

FRAME 1
resulting
(implicit) velocities

In total: the “box”,
under gravity + collision

- had rotated
- gained angular velocity (will keep rotating by inertia)

even the system does not (explicitly) handle rotations or
angular velocities

(works in 3D as well!)

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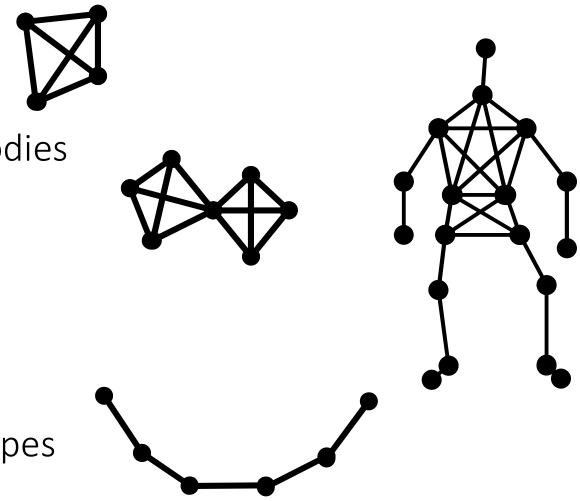
We can combine equidistance constraints to obtain…

- Rigid bodies
- Articulated bodies
- Ragdolls
- Cloth
- Non-elastic ropes
- …and more



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Positional constraint (in general terms)

- A predicate defined on the position(s) of a number of particles (usually, a small number: 1 - 4)

$$\mathcal{C}: (\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \ldots) \rightarrow \{ \, true \, , false \, \}$$

- For example, the equidistance constraints is

$$\mathcal{C}(\mathbf{p}_a, \mathbf{p}_b) \Leftrightarrow \|\mathbf{p}_a - \mathbf{p}_b\| = k_{CONST}$$

- They can be an equality ($=$) or an inequality ($\leq$ or $\geq$)

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Equality positional constraint: examples

- Equidistance constraint (the one we have seen):
  «these N particles must stay at distance k»
  - E.g: because they are linked my a metal rod of length k
- Fixed positions:
  «this particle must stay in position $\mathbf{p}_\mathrm{a}$ *»*
  - the particle is "pinned" in position
  - trivial to impose, but still useful!
- Coplanarity / collinearity:
  «these N particles must stay on a line / on a plane»

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Equality positional constraint: other examples

- Volume preservation:
  "The volume delimited by the squishy ballon defined by these particles is a constant k_{CONST} *" (e.g. because it's filled with water)*
- How to impose it:
  1. Estimate current total volume v
  2. uniform scale the entire object by factor $\sqrt[3]{f_{\mathrm{CONST}} / v}$ around its barycenter

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Inequality positional constraints: example

- "please don't sink below the ground"
  assuming the ground is the plane Y = 0

$$C(\mathbf{p}_a) \Leftrightarrow \mathbf{p}_a.y \geq 0$$

- Trivial to impose:
  just set the y to 0, if it is < 0

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Inequality positional constraints: example

- "this particle must stay above this fixed (and arbitrary) plane"
  - For example, because the plane is a solid unmovable slab
  - The plane is given by a point on it $\mathbf{p}_q$ and its normal $\hat{n}_q$

$$C(\mathbf{p}_a) \Leftrightarrow (\mathbf{p}_a - \mathbf{p}_q) \cdot \hat{n}_q \geq 0$$

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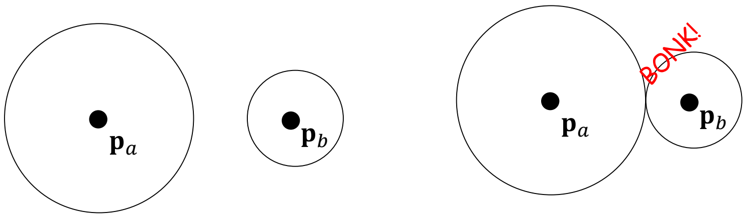
Inequality positional constraints: example

- These two particles must be at least k_{CONST} apart

  $$\mathcal{C}(\mathbf{p}_a, \mathbf{p}_b) \Leftrightarrow \|\mathbf{p}_a - \mathbf{p}_b\| \geq k_{CONST}$$

  - For example, because they are the centers of two rigid spheres and k_{CONST} is the sum of their radii
  - part of collision handling (see next lecture)



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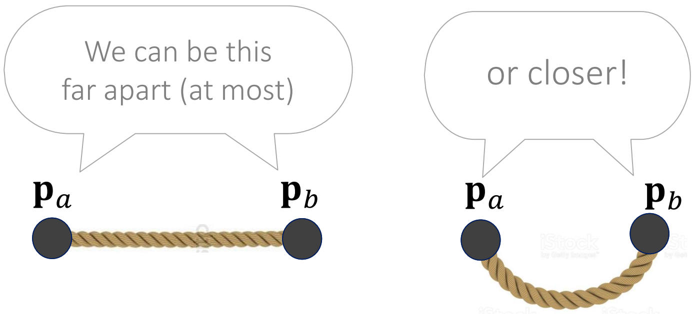
Inequality positional constraints: example

- These two particles must be at most k_{CONST} apart

  $$\mathcal{C}(\mathbf{p}_a, \mathbf{p}_b) \Leftrightarrow \|\mathbf{p}_a - \mathbf{p}_b\| \leq k_{CONST}$$

  - For example, because they are tied by an inextensible rope that has length k_{CONST} (but can fold)



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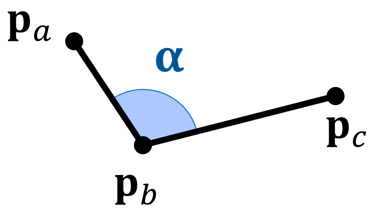
Inequality positional constraints: example

- Angle constraints, e.g. $\boldsymbol{\alpha} < \boldsymbol{\alpha}_{\max}$
  with $\boldsymbol{\alpha}$ the angle between $\mathbf{p}_a$, $\mathbf{p}_b$ and $\mathbf{p}_b$, $\mathbf{p}_c$
  - e.g., on joints: *«elbows cannot bend backward»*
  - (a constraint between three particles!)



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Enforcing one positional constraint (in general terms)

- Inequality constraint:
  1. *Test:* does the inequality already hold?
  2. If so: do nothing
  3. If not: enforce it as an equality (=) instead (see below!)
- Equality constraint:
  - All involved particles must be displaced
    from that current position, so that it now holds
  - There can be infinite ways to achieve this!
    Question: Which one to pick?

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Enforcing one equality constraint:
(assuming for now all particles have same mass)

- Answer:
  minimize the sum of *squared* displacements
  (with respect to current position)
- For each kind of constraint, we need to find the minimizer analytically
  ("analytically" = in closed form = "with formulas" = "solving a simple math problem on paper")
  - That's what we did for the equality constraint

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Enforcing one equality constraint
(assuming for now all particles have same mass)

- We want to enforce a constraint $\mathcal{C}$ on particles $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$ currently in positions $\mathbf{p}_\mathrm{a}, \mathbf{p}_\mathrm{b}, \mathbf{p}_\mathrm{c}, \ldots$

$$\mathcal{C}: (\mathbf{p}_\mathrm{a}, \mathbf{p}_\mathrm{b}, \mathbf{p}_\mathrm{c}, \ldots) \rightarrow \{\, true\,, false \,\}$$

- We must apply the displacements $\vec{d_\mathrm{a}}, \vec{d_\mathrm{b}}, \vec{d_\mathrm{c}}$ that are the

$$\underset{\vec{d_\mathrm{a}}, \vec{d_\mathrm{b}}, \vec{d_\mathrm{c}}, \ldots}{\operatorname{argmin}} \left(\|\vec{d_\mathrm{a}}\|^2 + \|\vec{d_\mathrm{b}}\|^2 + \|\vec{d_\mathrm{c}}\|^2 + \cdots \right)$$

$$\text{such that } \mathcal{C}\left(\mathbf{p}_\mathrm{a} + \vec{d_\mathrm{a}}, \mathbf{p}_\mathrm{b} + \vec{d_\mathrm{b}}, \mathbf{p}_\mathrm{c} + \vec{d_\mathrm{c}}, \ldots\right)$$

among all the choices that satisfy this,
we want the one which minimizes this

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Enforcing one equality constraint
(in general)

- We want to enforce a constraint $\mathcal{C}$ on particles a , b , c, ... in positions $\mathbf{p}_a$, $\mathbf{p}_b$, $\mathbf{p}_c$ and with masses m_a, m_b, m_c, ...

$$\mathcal{C}: (\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \dots) \rightarrow \{ true , false \}$$

- We must apply the displacements $\overrightarrow{d_a}$, $\overrightarrow{d_b}$, $\overrightarrow{d_c}$ found by:

$$\underset{\overrightarrow{d_a}, \overrightarrow{d_b}, \overrightarrow{d_c}, \dots}{\text{argmin}} \left(m_a \|\overrightarrow{d_a}\|^2 + m_b \|\overrightarrow{d_b}\|^2 + m_c \|\overrightarrow{d_c}\|^2 + \cdots \right)$$

$$\text{such that } \mathcal{C}\left(\mathbf{p}_a + \overrightarrow{d_a}, \mathbf{p}_b + \overrightarrow{d_b}, \mathbf{p}_c + \overrightarrow{d_c}, \dots\right)$$

among all the choices that satisfy this,
we want the one which minimizes this

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Example: solve the "please don't sink under this plane"

$$\text{C}(\mathbf{p}_a) \Leftrightarrow (\mathbf{p}_a - \mathbf{p}_q) \cdot \hat{n}_q \geq 0$$

a point on plane (const)
plane normal (const)

- We need to find displacement $\overrightarrow{d_a}$ as:

$$\underset{\overrightarrow{d_a}}{\text{argmin}} \left(m_a \|\overrightarrow{d_a}\|^2 \right)$$

$$\text{such that } \left(\mathbf{p}_a + \overrightarrow{d_a} - \mathbf{p}_q\right) \cdot \hat{n}_q \geq 0$$

- And the solution (in closed form) is...

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In pseudocode

```
Vector3 pa; // curr positions of a
float ma;   // mass (no effect here)
Vector3 pq; // point on the plane
Vector3 nq; // normal of the plane (unit)

Vector3 v = pa - pq;
float currDist = Vector3.dot( v , n );

if (currDist < 0.0)
   pa -= currDist * n; // just project!
else {} // constrain ok, nothing to do
```

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Example: the equidistance constraint (for unequal masses)

$$\mathcal{C}(\mathbf{p}_a, \mathbf{p}_b) \Leftrightarrow \|\mathbf{p}_a - \mathbf{p}_b\| = k_{CONST}$$

- With partcle masses m_a , m_b
- We need to the displacements $\overrightarrow{d_a}$, $\overrightarrow{d_b}$ found by minimizing:

$$\underset{\overrightarrow{d_a}, \overrightarrow{d_b}}{\operatorname{argmin}} \left(m_a \left\|\overrightarrow{d_a}\right\|^2 + m_b \left\|\overrightarrow{d_b}\right\|^2 \right)$$
$$\text{such that } \left\|(\mathbf{p}_a + \overrightarrow{d_a}) - (\mathbf{p}_b + \overrightarrow{d_b})\right\| = k_{CONST}$$

- And the solution (in closed form) is…

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Example: the equidistance constraint (for unequal masses)

```
Vector3 pa, pb; // curr positions of a,b
float ma, mb;   // masses of a,b
float d;        // distance (to enforce)

Vector3 v = pa - pb;
float currDist = v.length;

v /= currDist;  // normalization of v

float delta = currDist - d ;

/* solutions of the minimization: */
pa += ( mb/(ma+mb) * delta) * v;
pb -= ( ma/(ma+mb) * delta) * v;
```

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Observe and verify

- The way we have seen to impose…
  - The “fixed position” constraint
  - The “equidistance” constraint
  - The “stay above ground” constraint
  - Etc.

  are the ones that minimizes the mass-weighted squared displacements of the particles
  - (the mass is not always relevant)

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Position Based Dynamics (PBD) summary

- A general approach for computing dynamics
- Ingredients:
  1. Use Verlet integration **on particles**
     - their velocities are implicit
     - changes in positions induce changes in velocities
  2. Implement positional constraints **on particles** (e.g., equidistance constraint) to model things like:
     - Rigid bodies (their rotational speed is an emerging feature!)
     - Articulated / non rigid bodies
     - Basic collision detection

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Not forces: summary

$$\ldots \quad \vec{f} = \text{function}(\mathbf{p}, \ldots) \quad \ldots$$

not in here

- We have seen many kinds of real-world forces that are modelled by things that aren't "forces" in our simulation:
  - Frictions
    - *In reality:* a ("dissipative") force contrasting motion
    - Can be simulated by: drag / velocity damp
  - Violent sudden events, such as impacts
    - *In reality:* a very strong force that is sustained for a very short time << *dt*
    - E.g.: hitting a ball with a mace
    - Must be simulated by: impulses
  - Resistance forces
    - *In reality:* a force that contrast and nullifies an external force (e.g. gravity)
    - E.g.: what prevents your computer from falling through the table RN
    - Can be simulated by: positional constraints

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