Course Plan

- lec. 1: Introduction
- lec. 2: Mathematics for 3D Games
- lec. 3: Scene Graph
- lec. 4: Game 3D Physics +
- lec. 5: Game Particle Systems
- lec. 6: Game 3D Models
- lec. 7: Game Textures
- lec. 9: Game Materials
- lec. 8: Game 3D Animations
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- lec. 13: Rendering Techniques for 3D Games

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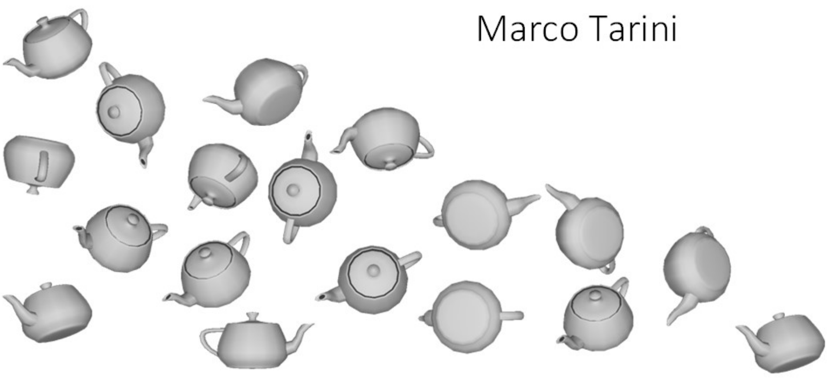
3D Videogames

Rotations: exercises

Marco Tarini



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2D rotations as 3D rotations

- A 2D rotation (of an angle α , around the origin) can be seen as the *restriction* of a 3D rotation in the X-Y plane (of an angle α , around the... Z axis!)
- Find this 3D rotation in *all* representations:
  - as... a 3x3 Matrix: $\begin{bmatrix} +\cos(\alpha) & -\sin(\alpha) & 0 \\ +\sin(\alpha) & +\cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - as... Axis-times-Angle: $[0,0,\alpha]$
  - as... Euler angles (Roll=Z, Pitch=X, Yaw=Y): $[\alpha,0,0]$
  - as... a quaternion: $\left[0,0,\sin\left(\frac{\alpha}{2}\right),\cos\left(\frac{\alpha}{2}\right)\right]$

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Exercises: *find the rotation that…*

- For all the following exercises: we can pick any rotation representation! (unless otherwise specified)
  - As long as we have algorithms to translate one representation into another
  - For each task, try to identify which format is the most convenient to use!

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Find the «from-to» rotation

- Problem: given a pair of versors $\hat{v}$ and $\hat{w}$, ($\hat{v}$ = *from* and $\hat{w}$ = *to*) find the minimal rotation that brings $\hat{v}$ into $\hat{w}$
  - useful problem in several contexts

minimal angle

e.g. AI aiming a bazooka

- A solution as axis-and-angle
  - the axis *a* is found as $\hat{v} \times \hat{w}$ (renormalizing it)
  - about the angle α , we know that its cosine is ($\hat{v} \cdot \hat{w}$) and its sine is $\| \hat{v} \times \hat{w} \|$, so $\alpha = \text{atan2}(\| \hat{v} \times \hat{w} \| \ , \ \hat{v} \cdot \hat{w})$

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Find the «look-at» rotation

- Given observer's position $\mathbf{e}$ and observed point $\mathbf{t}$ find the rotation (i.e., the orientation) for a character who must be looking in that direction
- That specification is incomplete: we also need another input: a «target up-vector» $\hat{\boldsymbol{u}}$
  - the character wants to keep its up-direction as similar as possible to $\hat{\boldsymbol{u}}$, while looking toward $\mathbf{t}$
  - Usually, the (world) up-vector, e.g. (in Unity) (0,1,0)
- Useful for… characters heads looking at something / facing toward something, setting up the camera…

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Find the «look-at» rotation

- Solution: as a 3×3 matrix
  - find the $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ directions of this local character
  - they must be 3 reciprocally orthogonal versors
  - they are the columns of the sought matrix
- that is (assuming Unity conventional axis names):
  - $\hat{\boldsymbol{z}} = (\mathbf{t} - \mathbf{e}) / ||\mathbf{t} - \mathbf{e}||$
  - $\hat{\boldsymbol{y}} = \hat{\boldsymbol{u}}$? Wrong: it wouldn't be necessarily orthogonal with $\hat{\boldsymbol{z}}$
  - but, $\hat{\boldsymbol{x}} = \hat{\boldsymbol{u}} \times \hat{\boldsymbol{z}} \;/\; || \hat{\boldsymbol{u}} \times \hat{\boldsymbol{z}} ||$ (note the re-normalization) because the right direction is orthogonal to both $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{u}}$
  - finally, $\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}}$

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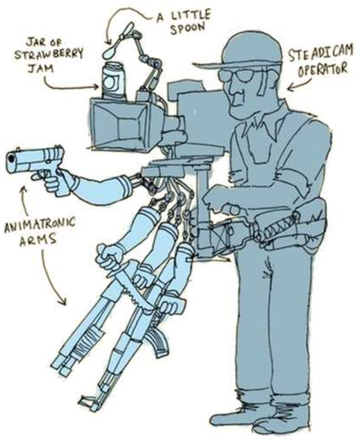
What about the "look-at" complete transform

- Setting up the complete transform of a camera (from the same data):
  - Camera position: is the translation component
  - the "look-at" rotation : is the rotation component
  - (scale component = 1)

In Computer Vision the set of these parameters are defined as the camera extrinsic parameters



"Camera-man in videogame logic"
unknown artist, circa 2010

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Update the orientation of a rolling ball *

- A ball with radius r stands on a flat plane (with plane normal $\hat{\mathrm{n}}$), and it's currently oriented with rotation R_0 and positioned (center position) in $\mathbf{p}_0$
- It then rolls in position $\mathbf{p}_1$ (staying on the plane)
- Find its new orientation R_1

Marble Madness, Atari, 1986

* a classic of many 3D games!
Including early "3D" games

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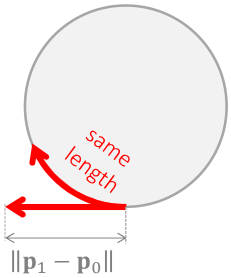
Update the orientation of a rolling ball *

Solution (trace): as axis-angle…



- The axis must be:
  - parallel to the ground; therefore, orthogonal to $\hat{\mathrm{n}}$!
  - orthogonal to the direction of motion $(\mathbf{p}_1 - \mathbf{p}_0)$
  - (also, it must be expressed as a unit vector)
- The angle α must satisfy…
  full-circumference : length-of-arch = full-circle : α
  ($2\,\pi\,r$; $\|\mathbf{p}_1 - \mathbf{p}_0\|$; $2\,\pi$ radiants, or 360°)

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Marco Tarini
Unviersità degli studi di Milano

Update the orientation of a rolling ball *

- Once we found the new motion R_N (rotation as a motion, a spin), in must be cumulated with the old state R_0 (rotation as a state, an orientation)

$$\mathrm{R}_0 \longleftarrow \mathrm{R}_N \circ \mathrm{R}_0$$

- To do so, it's more convenient to convert both into a format that allows for easy cumulation
  - Like quaternions
  - How to do so? See exercises at the end

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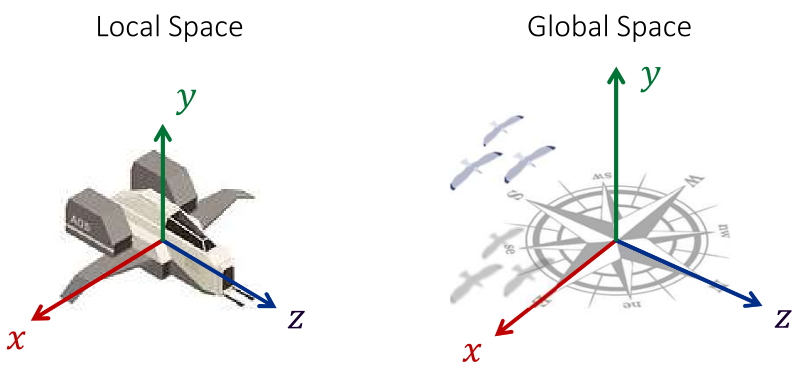
Find the orientation of a spaceship/airplane "character"



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Find the orientation of a spaceship/airplane “character”

- Find the orientation $\mathrm{R_P}$ of an airplane at spawn time
  - The airplane is going NNE, and climbing up at 30° angle.
  - Its wing lines is parallel to the ground.
- Local space of airplane:
  - X-axis: left-right (the direction of the wings)
  - Y-axis: below to above
  - Z-axis: engine-to-propeller
- World space:
  - X-axis: west to east
  - Y-axis: ground to sky
  - Z-axis: south to north

NNE = halfway between North and NE

(which handedness is world and local spaces?)

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Find the orientation of the head of the pilot of previous exercise

- The head of the pilot inside that plane is tilted 20° to the left, and 10° degrees above
- What it is its orientation $\mathrm{R_H}$?
- Local space of the head:
  - X-axis: left-eye to right-eye
  - Y-axis: chin to top of the head
  - Z-axis: view direction

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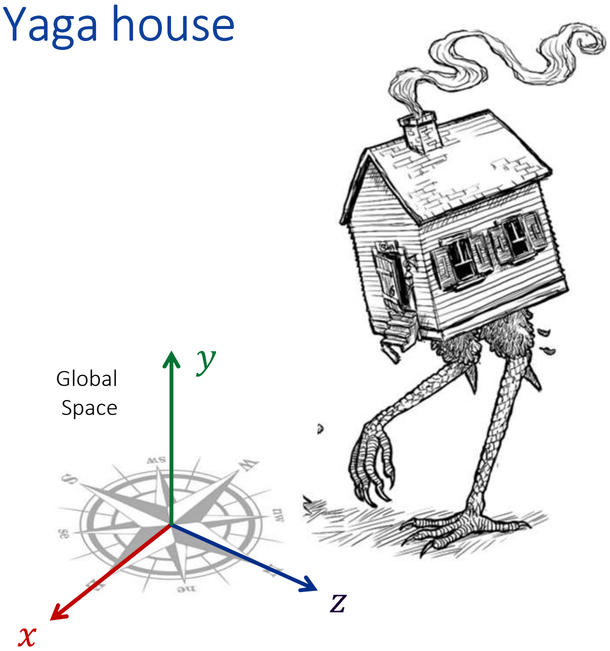
Baba Yaga house



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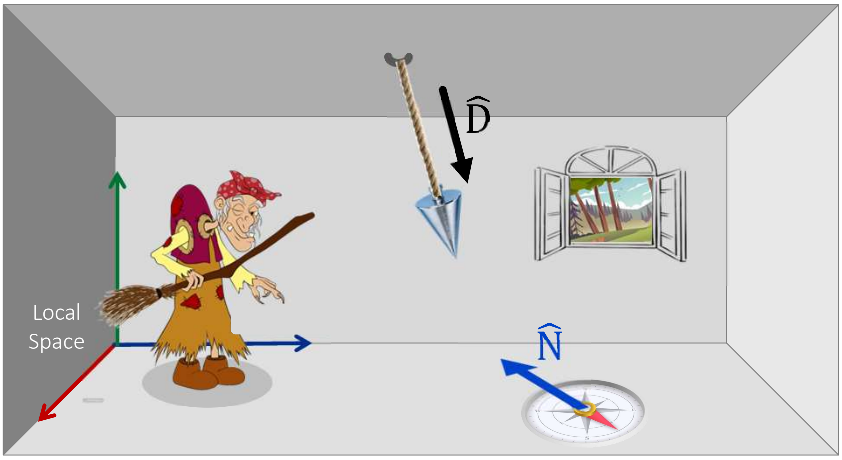
Baba Yaga house: internal view.

Which rotation is associated to the house?



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Baba Yaga house: internal view.
Which rotation is associated to the house?

- The witch Baba Yaga lives on a moving house on chicken leg
- The hose is now sitting still with some orientation.
- Inside the house, Baba Yaga
keeps a compass (ita: *"una bussola"*)
and a plumb line (ita: *"un filo a piombo"*)
- Knowing the current direction of the plumb line $\hat{D}$ is pointing
(which points downward)
and the direction of the compass needle $\hat{N}$
(parallel to the house floor, and always pointing North)
can you express the orientation of the house? (in any form?)
- Note: Baba Yaga measures $\hat{D}$ and $\hat{N}$ as versors in *house space*
(the local space of the house)
(where, for example axis Y is orthogonal to the floor)

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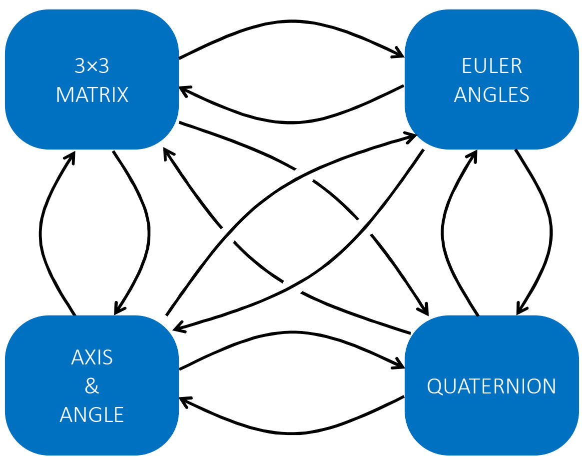
Converting between rotation representations: exercises



Every representation
is useful in some
circumstance.

In a game engine or lib,
conversions are useful.

Not all arrows
need to be implemented!

Just ensure there is a path
from anything to anything.

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From: axis-&-angle
To: quaternion, or viceversa

- Trivial exercise. Observations:
  - When going from an angle-based representation (*Euler angles*, *Axis-&-Angle*) to a non-angle-based representation (*Matrix*, *Quaternion*) you'll need trigonometric functions (**sin** , **cos** , …)
  - When going from a non-angle-based representation (*Euler angles*, *Axis-&-Angle*) to an angle-based representation (*Matrix*, *quaternion*) you'll need inverse trigonometric functions (**asin** , **acos** , **atan2**) ← Remember this convenient one exists!

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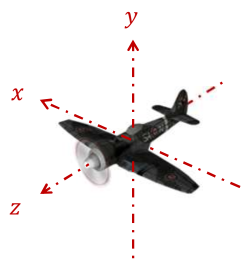
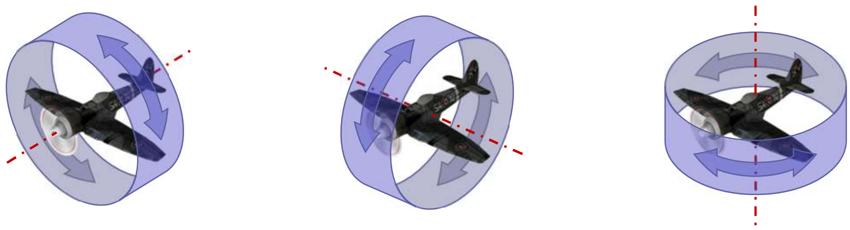
from: Euler angles
to: 3x3 matrix

- Question:
  - Which matrix $\mathbf{R}$ does this?



Assume "Unity" axis definitions



Roll α (1st) | Pitch β (2nd) | Yaw γ (3rd)

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from: Euler angles
to: 3x3 matrix

the order is prescribed by the choice of Euler Angles

$$R = R_y(\gamma) \cdot R_x(\beta) \cdot R_z(\alpha)$$

$$\begin{bmatrix} +\cos(\beta) & 0 & +\sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & +\cos(\beta) \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\cos(\alpha) & -\sin(\alpha) \\ 0 & +\sin(\alpha) & +\cos(\alpha) \end{bmatrix} \quad \begin{bmatrix} +\cos(\alpha) & -\sin(\alpha) & 0 \\ +\sin(\alpha) & +\cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

See: rotations in 2D

- What about the vice-versa?
  - a more difficult exercise
  - requires inverse trigonometric functions (of course)

from: Euler angles - to: Quaternion

- Which quaternion encodes the rotation by Euler Angles (ROLL, PITCH, YAW) = $(\alpha_R, \alpha_P, \alpha_Y)$?

Rotations:

name	Axis	Order	Axis & Angle	As quaternion		
ROLL	Z	1st	$\begin{pmatrix}0\\0\\1\end{pmatrix}, \alpha_R$	$s_R\ k + c_R$	with	$s_R = \sin\left(\frac{\alpha_R}{2}\right)$, $c_R = \cos\left(\frac{\alpha_R}{2}\right)$
PITCH	X	2nd	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \alpha_P$	$s_P\ i + c_P$	with	$s_P = \sin\left(\frac{\alpha_P}{2}\right)$, $c_P = \cos\left(\frac{\alpha_P}{2}\right)$
YAW	Y	3rd	$\begin{pmatrix}0\\1\\0\end{pmatrix}, \alpha_Y$	$s_Y\ j + c_Y$	with	$s_Y = \sin\left(\frac{\alpha_Y}{2}\right)$, $c_Y = \cos\left(\frac{\alpha_Y}{2}\right)$

Answer: $(s_Y\ j + c_Y)(s_P\ i + c_P)(s_R\ k + c_R) =$

$(s_Y\ j + c_Y)(-s_P\ s_R\ j + s_P c_R\ i + \ c_P\ s_R\ k + c_P c_R) \ =$

$\ldots$

from: 3x3 matrix to: axis-&-angle

- Question:
  - Given a rotation matrix R , find axis $\hat{\mathrm{a}}$ and rotation angle α
  - Assumption: R is actually a rotation matrix
- Trace:
  1. Observation: for the given matrix R , $\mathrm{R}\,\hat{\mathrm{a}} = \hat{\mathrm{a}}$ (why?)
  2. In other words, $\hat{\mathrm{a}}$ is an eigenvector of R of eigenvalue 1
  3. Find α : remember atan2 exists

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from: axis-&-angle to: 3x3 matrix

- Question:
  - Which matrix R rotates by α degrees around axis $\hat{\mathrm{a}}$?
- Trace:
  1. Find any rotation matrix R_A mapping $\hat{\mathrm{a}}$ the axis into the X axis (hint: find three orthogonal versors to use as columns of R_A , one of them being $\hat{\mathrm{a}}$)
  2. Define a rotation matrix R_x rotating by α around X axis
  3. Then: $\mathrm{R} = \mathrm{R}_A^{-1} \cdot \mathrm{R}_x \cdot \mathrm{R}_A$ (understand why)

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from: Quaternion to: 3x3 Matrix

- Which matrix $\mathbf{M}$ encodes the same rotation as quaternion $\mathbf{q} = (a\,i + b\,j + c\,k + d)$?
- Trace: let's find out the three columns $\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}$ of $\mathbf{M}$ ...

Axis (versors)	In local space			In global Space			
	in cartesian coords		as quaternion		as quaternion		in cartesian coords	
X	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	➡	i	➡	$\mathbf{q}\,i\,\bar{\mathbf{q}} = \ldots$	➡	$\begin{pmatrix}?\\?\\?\end{pmatrix}$	➡ $\mathbf{c_0}$
Y	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	➡	j	➡	$\mathbf{q}\,j\,\bar{\mathbf{q}} = \ldots$	➡	$\begin{pmatrix}?\\?\\?\end{pmatrix}$	➡ $\mathbf{c_1}$
Z	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	➡	k	➡	$\mathbf{q}\,k\,\bar{\mathbf{q}} = \ldots$	➡	$\begin{pmatrix}?\\?\\?\end{pmatrix}$	➡ $\mathbf{c_2}$

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A "trivial" final exercise: space transformation of vectors

- Let $\vec{\mathbf{v}}$ be a vector defined in local space
- Let $\vec{\mathbf{w}}$ be the corresponding vector in world space
- Let $\hat{\boldsymbol{a}}_X\ \hat{\boldsymbol{a}}_Y\ \hat{\boldsymbol{a}}_Z$ be the three versors describing the three object-space axis (expressed in world space)
  - assume the transformation has scaling = 1
- Problem 1: given $\vec{\mathbf{v}}$, find $\vec{\mathbf{w}}$
- Problem 2: given $\vec{\mathbf{w}}$, find $\vec{\mathbf{v}}$
- Solutions: trivial, right?
  - The rotation of the transform is given by the matrix $\mathrm{R} = [\hat{\boldsymbol{a}}_X \mid \hat{\boldsymbol{a}}_Y \mid \hat{\boldsymbol{a}}_Z]$
  - Then, by definition, $\vec{\mathbf{w}} = \mathrm{R}\,\vec{\mathbf{v}}$ and $\vec{\mathbf{v}} = \mathrm{R}^{-1}\,\vec{\mathbf{w}}$
- However, the task is to address both problems using only geometric intuition, and the algebra of point, vector, versors…
  - without any concept of "rotation"

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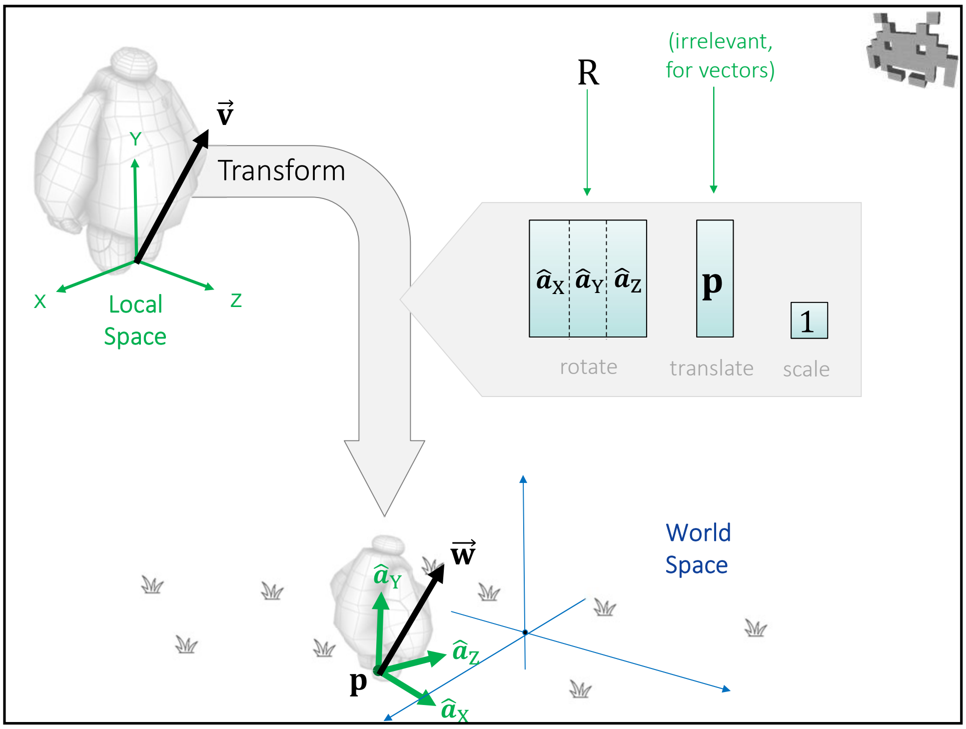

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Solution (trace)

$\vec{\mathbf{v}} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad \vec{\mathbf{w}} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$

- Problem 1: given $\vec{\mathbf{v}}$, find $\vec{\mathbf{w}}$
  - See how $\vec{\mathbf{w}}$ can be found as a linear combination of $\hat{\boldsymbol{a}}_X$, $\hat{\boldsymbol{a}}_Y$, $\hat{\boldsymbol{a}}_Z$
  - ...with linear weights v_x, v_y, v_z (the coordinates of $\vec{\mathbf{v}}$)
  - *Bonus:* rewrite that expression in matrix form...
    (that is, using $\vec{\mathbf{v}}$, not v_x, v_y, v_z)
- Problem 2: given $\vec{\mathbf{w}}$, find $\vec{\mathbf{v}}$
  - See how each of v_x, v_y, v_z (the coordinates of $\vec{\mathbf{v}}$)
    can be found as a dot product (...with $\hat{\boldsymbol{a}}_X$, $\hat{\boldsymbol{a}}_Y$, $\hat{\boldsymbol{a}}_Z$)
  - [note: this only works because $\hat{\boldsymbol{a}}_X$, $\hat{\boldsymbol{a}}_Y$, $\hat{\boldsymbol{a}}_Z$ are unitary and orthogonal]
  - *Bonus:* rewrite that expression in matrix form...
    (that is, using $\vec{\mathbf{v}}$, not v_x, v_y, v_z)
- ...you rediscovered that R and R^T are the inverse of each other!
  - as they are the ways to solve two *inverse* geometrical problems

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