3D videogames

Additional notes on dual-quaternions

(not part of the exam, but maybe useful in life)

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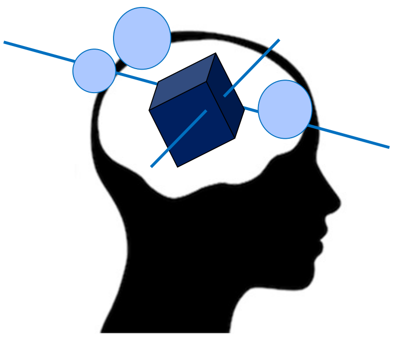

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Dual Quaternions: overview 1/2

- New "fantasy" assumption: there is a ε such that $\varepsilon \neq 0$, $\varepsilon^2 = 0$
- A dual quaternion: $p + \varepsilon\, q$, with $p,q \in \mathbb{H}$

quaternion set

- So, eight scalars (a,b,c,d,e,f,g,h)
  - weights for: $1, i, j, k, \varepsilon, \varepsilon i, \varepsilon j, \varepsilon k$

$$\underbrace{\overbrace{a}^{\text{real part of p}} + \overbrace{b\,i + c\,j + d\,k}^{\text{imaginary part of p}}}_{\substack{p\\\text{the "primal" quaternion}}} + \underbrace{\overbrace{e\,\varepsilon}^{\text{real part of q}} + \overbrace{f\,\varepsilon\,i + g\,\varepsilon\,j + h\,\varepsilon\,k}^{\text{imaginary part of q}}}_{\substack{\varepsilon\,q\\\text{the "dual" quaternion}}}$$

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Dual Quaternions: overview 2/2

- A dual quaternion $\overbrace{\mathrm{p}}^{a+b\,i+c\,j+d\,k} + \varepsilon\, \overbrace{\mathrm{q}}^{e+f\,i+g\,j+h\,k}$ can represent:
  - a point in 3D , when p = 1 and Real(q) = *e* = 0
    then Im(q) = (*f*,*g*,*h*) = (*x*,*y*,*z*)
  - a vector/versor in 3D , when p = 0 and Real(q) = *e* = 0
    then Im(q) = (*f*,*g*,*h*) = (*x*,*y*,*z*)
  - a roto-translation, when ||p|| = 1 and p·q = 0 (4D dot product)
    then p encodes the rotational part
    and q encodes the translational part
  - (nothing, otherwise)
- To roto-translate a point a with roto-trans b
  just "conjugate" their representations a' ⟵ b ∗ a ∗ $\overline{\mathrm{b}}$

dual quaternion multiplication

dual-quaternion conjugate: $\overline{\mathrm{p}} - \varepsilon\, \overline{\mathrm{q}}$

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Quaternion math: Dot Product (let's see a few rules that will be useful later)

- It's computed considering the quaternions as vectors in 4D
- For today, let's denote it as $\langle\, \mathbf{p} \,,\mathbf{q} \,\rangle$
  to avoid confusion with the standard quaternion product $\mathbf{p}\ \mathbf{q}$
- The dot can also be rewritten as the real part
  of the product of $\mathbf{p}$ with the conjugate of $\mathbf{q}$,
  or vice-versa, any order:
  $$\langle\, \mathbf{p} \,,\mathbf{q} \,\rangle = Re(\mathbf{p}\ \overline{\mathbf{q}}) = Re(\,\overline{\mathbf{q}}\ \ \mathbf{p}\,)$$
  Exercise: understand why (look at the formula of the product!)
- Dot product of a quaternion with itself:
  $$\langle\, \mathbf{p} \,,\mathbf{p} \,\rangle\ = \mathbf{p}\ \overline{\mathbf{p}} = \|\mathbf{p}\|^2$$
  Exercise: understand why, including why is the imaginary part 0
- Also: $(\mathbf{p} + \overline{\mathbf{p}}) = 2\, Re(\mathbf{p})$
  $(\mathbf{p} - \overline{\mathbf{p}}) = 2\, Im(\mathbf{p})$
  Exercise: understand why
- Also: $\overline{(\mathbf{p}\ \mathbf{q})} = \overline{\mathbf{q}}\ \ \overline{\mathbf{p}}$
  Exercise: verify!

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Dual Quaternion math: Product

dual quaternion multiplication

$$(\mathbf{p}_0 + \varepsilon\,\mathbf{q}_0) * (\mathbf{p}_1 + \varepsilon\,\mathbf{q}_1)$$
$$=$$
$$\mathbf{p}_0\,\mathbf{p}_1 + \varepsilon\,(\mathbf{p}_0\,\mathbf{q}_1 + \mathbf{q}_0\,\mathbf{p}_1) + \cancel{\varepsilon^2\,\mathbf{q}_0\,\mathbf{q}_1}$$

Naturally, it isn't commutative (or anticommutative), but it's associative.
Notation: we will always denote the dual-quat multiplication with $*$.
(observe that we won't need the dot product (in 8D) for dual-quats)

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Dual Quaternion math: Conjugate

$\mathbf{a} =$

	imaginary			*real*
primal	a	b	c	d
dual	e	f	g	h

$\bar{\mathbf{a}} =$

	imaginary			*real*
primal	$-a$	$-b$	$-c$	d
dual	e	f	g	$-h$

Rationale: conjugate both primal and dual quat, flip sign of dual quat: $\overline{\mathbf{p} + \varepsilon\,\mathbf{q}} = \overline{\mathbf{p}} - \varepsilon\,\overline{\mathbf{q}}$

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Dual Quaternions of pure rotations & translations

- Pure rotation dual quaternion by axis $\hat{a}$, angle α:

	imaginary	real
primal	$\sin\left(\frac{\alpha}{2}\right)\hat{a}$	$\cos\left(\frac{\alpha}{2}\right)$
dual	0	0

- Pure translation dual quaternion by vector $\vec{t}$:

	imaginary	real
primal	0	1
dual	$\frac{1}{2}\vec{t}$	0

Exercise: check that, in both cases...
- the primal has norm 1
- the primal dot the dual is 0

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Proof 1/2

$$\underbrace{\left(1+\varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}_{\text{the dual qual representing a pure translation by }\vec{t}} * \underbrace{\left(1+\varepsilon\left(\vec{v},0\right)\right)}_{\text{the dual qual representing a 3D point of coords }\vec{v}} * \underbrace{\left(1+\varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}_{\text{the conjugate of}} =$$

$$= \left(1+\varepsilon\left(\frac{1}{2}\vec{t},0\right)\right) * \left(1+\varepsilon\left(\vec{v}+\frac{1}{2}\vec{t},0+0\right)\right) =$$

$$= \underbrace{1+\varepsilon\left(\vec{v}+\frac{\cancel{2}}{\cancel{2}}\vec{t},0\right)}_{\text{dual-quat representing 3D point }\vec{v}+\vec{t}}$$

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Proof 2/2

$$\overbrace{(\mathbf{r}+\varepsilon\,0)}^{\text{dual-quat representing a pure rotation}} * \overbrace{\big(1+\varepsilon\,(\vec{\mathrm{v}},0)\big)}^{\text{dual-quat representing 3D point }\vec{\mathrm{v}}} * \overbrace{(\bar{\mathbf{r}}+\varepsilon\,0)}^{\text{conjugate of}} =$$

$$= (\mathbf{r}+\varepsilon\,0) * \overbrace{(\bar{\mathbf{r}}+\varepsilon\,(\vec{\mathrm{v}},0)\,\bar{\mathbf{r}})}^{} =$$

$$= \underbrace{\cancelto{1}{\mathbf{r}\,\bar{\mathbf{r}}} + \varepsilon\underbrace{(\mathbf{r}\,(\vec{\mathrm{v}},0)\,\bar{\mathbf{r}})}_{\text{quat representing the rotation }\vec{\mathrm{v}}'\text{ of }\vec{\mathrm{v}}}}_{\text{Dual-quat representing rotated 3D point }\vec{\mathrm{v}}'}$$

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Dual Quaternion for a roto-translation

$$\overbrace{(1+\varepsilon\,\mathbf{t})}^{\text{then, translate}} * \overbrace{(\mathbf{r}+\varepsilon\,0)}^{\text{first, rotate}}$$

$$=$$

$$\mathbf{r} + \varepsilon\,\mathbf{t}\,\mathbf{r}$$

		imaginary	*real*
with...	$\mathbf{t}$ =	$\frac{1}{2}\vec{\mathrm{t}}$	0
	$\mathbf{r}$ =	$s\,\hat{\mathrm{a}}$	c
and so...	$\mathbf{t}\ \mathbf{r} = \frac{1}{2}$	$s\ \vec{\mathrm{t}} \times \hat{\mathrm{a}} + c\,\vec{\mathrm{t}}$	$-s\,\hat{\mathrm{a}}\cdot\vec{\mathrm{t}}$

with...

$s = \sin\left(\frac{\alpha}{2}\right)$

$c = \cos\left(\frac{\alpha}{2}\right)$

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Conclusion: the dual quat for the rotation around $\hat{a}$ by α followed by trans by $\vec{t}$

	imaginary	*real*
primal	$s\ \hat{a}$	c
dual	$s\ \vec{v} \times \hat{a}\ +\ c\ \vec{v}$	$-s\ \hat{a} \cdot \vec{v}$

$$s = \sin\left(\frac{1}{2}\alpha\right)$$

$$c = \cos\left(\frac{1}{2}\alpha\right)$$

$$\vec{v} = \frac{1}{2}\vec{t}$$

Exercise: check that...

- the primal has norm 1
- the primal *dot* the dual is 0

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$$\overbrace{\left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}^{\text{the dual qual representing a pure translation by } \vec{t}} * \overbrace{\left(0 + \varepsilon\,(\vec{v},0)\right)}^{\text{the dual qual representing a 3D vector of coords } \vec{v}} * \overbrace{\left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right)}^{\text{the conjugate of}} =$$

$$= \left(1 + \varepsilon\left(\frac{1}{2}\vec{t},0\right)\right) * \overbrace{\left(0 + \varepsilon\left(\vec{v},0\right)\right)} =$$

$$= \underbrace{0 + \varepsilon\left(\vec{v},0\right)}_{\text{dual-quat representing 3D vector } \vec{v}}$$

Dual quat applied to vectors 1/2

(they DON'T translate!)

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$$\overbrace{(\mathbf{r}+\varepsilon\,0)}^{\text{dual-quat representing a pure rotation}} * \overbrace{\big(\boxed{0}+\varepsilon\,(\vec{\mathrm{v}},0)\big)}^{\text{dual-quat representing specifically a vector }\vec{\mathrm{v}}} * \overbrace{(\bar{\mathbf{r}}+\varepsilon\,0)}^{\text{conjugate of}} =$$

$$= (\mathbf{r}+\varepsilon\,0) * \overbrace{(0+\varepsilon\,(\vec{\mathrm{v}},0)\,\bar{\mathbf{r}})} =$$

$$= \underbrace{\boxed{0}+\varepsilon(\underbrace{\mathbf{r}\,(\vec{\mathrm{v}},0)\,\bar{\mathbf{r}}}_{\text{quat representing the rotation }\vec{\mathrm{v}}'\text{ of }\vec{\mathrm{v}}})}_{\text{Dual-quat representing }\vec{\mathrm{v}}'}$$

Dual quat applied to vectors 2/2

(they rotate!)

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Inverting a Dual Quaternion roto-translation: conjugating both primal and dual quaterions

Proof:

$$\overbrace{(\mathbf{r}+\varepsilon\,\mathbf{t})}^{\text{a dual quat representing a roto-translation}} * \overbrace{(\bar{\mathbf{r}}+\varepsilon\,\bar{\mathbf{t}})}^{\text{the dual-quat representing the inverse}}$$

$$=$$

$$\underbrace{\mathbf{r}\,\bar{\mathbf{r}}}_{\substack{1\\ \text{because}\\ \mathbf{r}\\ \text{is unitary}}} + \;\varepsilon\,(\underbrace{\mathbf{t}\,\bar{\mathbf{r}}+\mathbf{r}\,\bar{\mathbf{t}}}_{\substack{\mathbf{t}\,\bar{\mathbf{r}}\text{ plus its conjugate (why?)}\\ \text{that is, twice}\\ \text{the real part of }\mathbf{t}\,\bar{\mathbf{r}}\text{ (why?)}\\ \text{that is, twice}\\ <\mathbf{t}\,,\,\mathbf{r}>\\ \text{that is,}\\ 0\\ \text{because it's}\\ \text{a roto-translation}}})$$

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Dual quaternions as rototranslation (summary of other operations)

- Each roto-translation is expressed by exactly two (opposite!) dual quaternions
- Their primal is unitary, their dual is orthogonal to primal
- Dual quaternion 1 (that is, 1 + ε 0) is the identity (as so is -1)
- Cumulation: multiplication (second $*$ first)
- Inversion: conjugation of both primal and dual (not the same as: dual-quat conjugation)

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Extracting the translation of a Dual Quaternion

Let's just see where the origin goes:

$$\overbrace{(\mathbf{r} + \varepsilon\,\mathbf{t})}^{\text{a dual quat representing a roto-translation}} * \overbrace{\cancel{1}}^{\text{the dual quat representing the origin}} * \overbrace{(\bar{\mathbf{r}} - \varepsilon\,\bar{\mathbf{t}})}^{\text{the conjugated of}}$$

$$=$$

$$\underbrace{\mathbf{r}\,\bar{\mathbf{r}}}_{\substack{1 \\ \text{because } \mathbf{r} \\ \text{is unitary}}} + \varepsilon\,\underbrace{(\mathbf{t}\,\bar{\mathbf{r}} - \mathbf{r}\,\bar{\mathbf{t}})}_{\substack{\mathbf{t}\,\bar{\mathbf{r}} - \overline{\mathbf{t}\,\bar{\mathbf{r}}} \\ \text{therefore twice the} \\ \text{imaginary part of } \mathbf{t}\,\bar{\mathbf{r}}}}$$

$$=$$

$$1 + \varepsilon\,\underbrace{2Im(\,\mathbf{t}\,\bar{\mathbf{r}}\,)}_{\text{the 3D vector that we are looking for}}$$

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Dual Quaternion for a .... "translo-rotation" :-D
(compare with roto-translation!)

$$\underbrace{(\mathbf{r} + \varepsilon\, 0)}_{\text{then, rotate}} * \underbrace{(1 + \varepsilon\, \mathbf{t})}_{\text{first, translate}}$$

$$=$$

$$\mathbf{r} + \varepsilon\, \mathbf{r}\, \mathbf{t}$$

		imaginary	real
with...	$\mathbf{t}$ =	$\frac{1}{2}\vec{t}$	0
	$\mathbf{r}$ =	$s\,\hat{a}$	c
and so...	$\mathbf{t}\,\mathbf{r} = \frac{1}{2}$	$s\ \hat{a} \times \vec{t} + c\,\vec{t}$	$-s\ \hat{a} \cdot \vec{t}$

$s = \sin\left(\frac{\alpha}{2}\right)$

$c = \cos\left(\frac{\alpha}{2}\right)$

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Interpolating Dual Quaternions

$$\text{mix}(\ \mathbf{p}_0 + \varepsilon\, \mathbf{q}_0\ ,\ \mathbf{p}_1 + \varepsilon\, \mathbf{q}_1\ ,\ t\ )$$

1. Take shortest path:
   if $\langle\, \mathbf{p}_0\, ,\ \mathbf{p}_1 \rangle$ negative, then flip *both* $\mathbf{p}_1$ and $\mathbf{q}_1$
2. Interpolate *both* primal & dual (LERP):
   $\mathbf{p} = \text{mix}(\ \mathbf{p}_0\ ,\ \mathbf{p}_1\, ,\ t\ )$
   $\mathbf{q} = \text{mix}(\ \mathbf{q}_0\ ,\ \mathbf{q}_1\, ,\ t\ )$
3. Re-enforce $\mathbf{p}$ to be unitary:
   divide *both* $\mathbf{p}$ and $\mathbf{q}$ by $\|\mathbf{p}\|$
4. Re-enforce $\mathbf{q}$ to be orthogonal to $\mathbf{p}$:
   subtract $\langle\, \mathbf{p}\, , \mathbf{q}\, \rangle\, \mathbf{p}$ from $\mathbf{q}$ (why?)

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