3D video games

3D Game Physics

Marco Tarini

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Course Plan

- lec. 1: Introduction
- lec. 2: Mathematics for 3D Games
- lec. 3: Scene Graph
- lec. 4: Game 3D Physics +
- lec. 5: Game Particle Systems
- lec. 6: Game 3D Models
- lec. 7: Game Textures
- lec. 9: Game Materials
- lec. 8: Game 3D Animations
- lec. 10: 3D Audio for 3D Games
- lec. 11: Networking for 3D Games
- lec. 12: Artificial Intelligence for 3D Games
- lec. 13: Rendering Techniques for 3D Games

computer animation

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Animation in games

but, a note on terminology:
in some contexts, procedural means
"produced by a *simple* procedure"
as opposed to "physically simulated"

Non procedural	Procedural
Assets!	Physics engine
Fully controlled by artist/designer (dramatic effects!)	Less control
Realism: depends on artist's skill	Physics-driven realism
Does not adapt to context	Auto adaptation to context
Repetition artefacts	Naturally repretition free

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Physics simulation in videogames

- 3D, or 2D
- "soft" real-time
- efficiency
  - 1 frame = 33 msec (at 30 FpS)
  - physics = 5% - 30% max of computation time
- plausibility
  - but not necessarily *accuracy*
- robustness
  - should almost never "explode"
  - it's tolerable to have inconsistencies over a few frames, as long as it recovers in subsequent frames

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Physics in games: cosmetics or gameplay?

- Just a graphic accessory? (for realism!)
  - e.g.:
    - particle effects (w/o feedback)
    - secondary animations
    - Ragdolling
- Or a gameplay component?
  - e.g. physics based puzzles
  - Popular approach in 2D (since always!)

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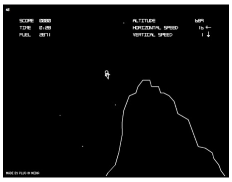

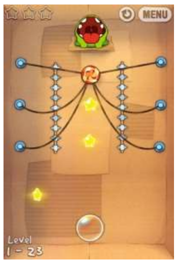

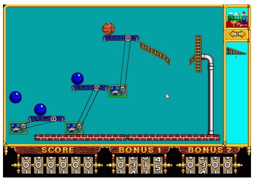

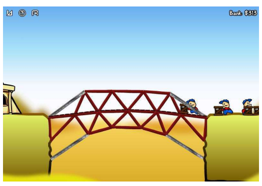

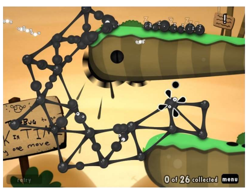

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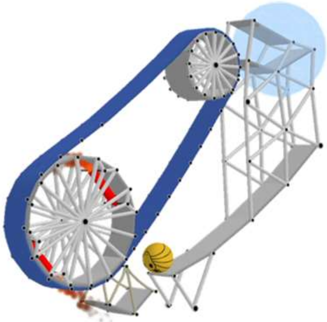
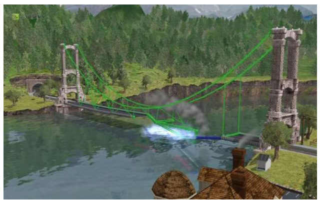
- Just a graphic accessory? (for realism!)
  - e.g.:
    - particle effects (w/o feedback)
    - secondary animations
    - Ragdolling
- Or a gameplay component?
  - e.g. physics based puzzles
  - Rising trend in 3D





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Physics engine: intro

- Game engine module
  - executed in real time at game run-time
- A high-demanding computation
  - on a very limited time budget!
- ...but highly parallelizable
  - potentially, highly parallel

==> good fit for hardware support

(just like the Rendering Engine)

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Hardware for Physics engine

To exploit a strong parallelism, you need a strongly parallel hardware!

- For a brief moment ~2006: PPU
  - "Physics Processing Unit"
  - HW unit specialized for physics
- After that: GP-GPU
  - "General Purpose Graphics Processing Unit"
    = Use of the graphics card for generic tasks
    (not related with 3D computer graphics)
  - or, Cuda (nVidia), OpenCL (openSource)

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Main Software (libraries, SDK)

havok	mostly CPU (Microsoft)
PhysX	CPU+GPU (CUDA) NVidia
Bullet Physics Library	open source, free, HW accelerated (OpenCL) + CPU
Open Dynamics Engine	open source, free
Box2D	2D, open source, free

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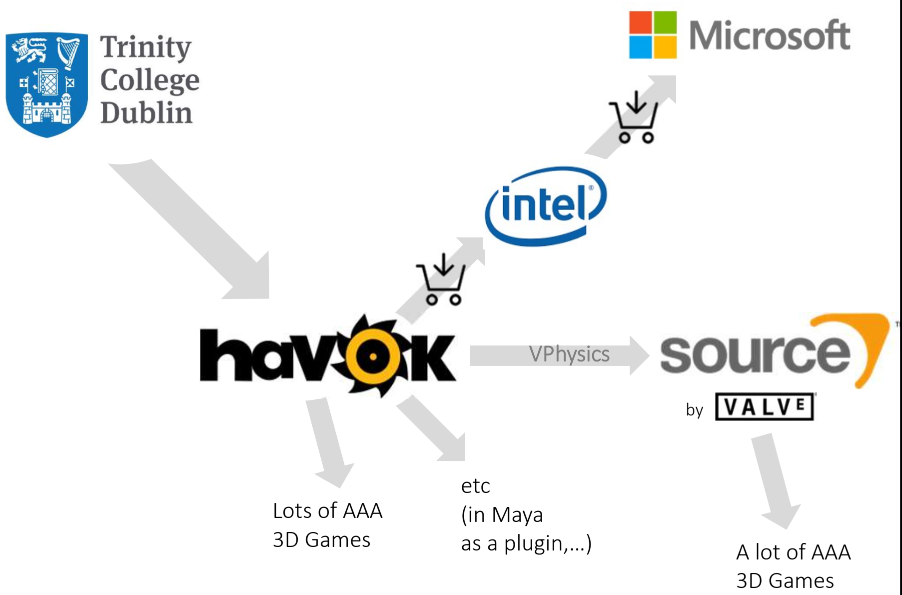
Brief history



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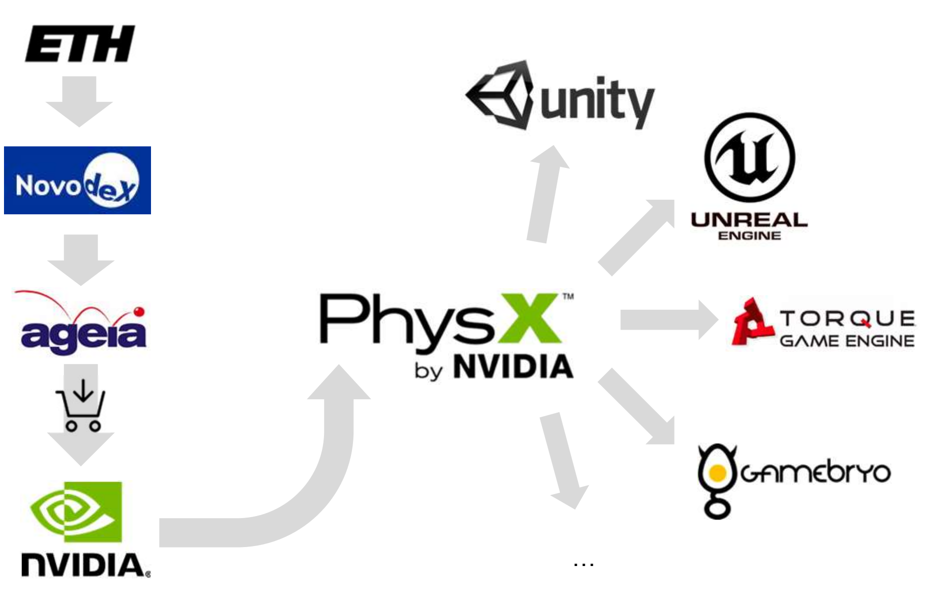
Brief history



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The 2 tasks of the Physics engine

1. Dynamics (Newtonian)
for objects such as:
- Particles
- Rigid bodies
- Articulated bodies
  - e.g. "ragdolling"
- Soft bodies
  - Ropes (specific solutions)
  - Cloth (specific solutions)
  - Hair (specific solutions)
  - Free-form deformation bodies (general)
- Fluids
  - Expensive!

2. Collision handling
- Collision detection
- Collision response

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Fields of study

Dynamics
The motion,
as a result of **forces**

Example:
"Subject to gravity, how will this pendulum swing?"

Statics
Equilibrium states,
minimal **energy** states

Example:
"In which state(s) can this pendulum be still?"

Kinematics
The **motion** itself, no matter why it moves

Example:
"If the angular speed of the pendulum is currently X, how fast is the ball moving?" (or vice versa)

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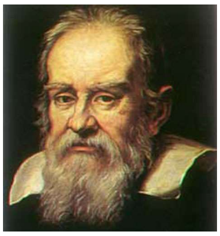
Newtonian Dynamics



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Physics and spaces (observation)

- The scene hierarchy (the scene graph), and the entire distinction between local and global space, its's entirely “in our mind”
  - It's a useful abstraction to control or code *scripted* animations
  - E.g., kinematics animations, skeletal animations…
- But physics *doesn't care* about any of it
  - Physics happens entirely in global (world) space
  - Persistent spatial relationships (e.g., between a car and its wheels) either exists due to physical constraints, or they are irrelevant
  - Even if they physically exists, they are still enforced in global space, like all the rest of the physics simulation
  - Physics simulation computes changes to objects states (position, orientation…) in global space
  - But, as we know, these updates can be converted/stored in local space

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Spatial placement of a (rigid) object

2D Physics

- Position:
  (x,y)
- Orientation:
  (α) – angle (scalar)

3D Physics

- Position:
  (x,y,z)
- Orientation:
  quaternion or
  axis,angle or
  axis * angle or
  3x3 matrix or
  Euler angles

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Newtonian dynamics: summary

Current object placement	Rate of change of ← (d / dt)	← "with mass" (momentum)	What changes the rate of change (d^2 / dt^2)	← "with mass"
Position p p = (x,y,z)	Velocity $\vec{v}$ $\vec{v} = \dot{p}$ ($\|\vec{v}\|$ = "speed")	Momentum $m\,\vec{v}$	Acceleration $\vec{a} = \dot{\vec{v}} = \ddot{p}$	Force $\vec{f}$ $\vec{f} = m\,\vec{a}$
Orientation (e.g. quaternion)	Angular velocity $\vec{\omega}$	Angular momentum $I\,\vec{\omega}$ I = moment of inertia	Angular acc. $\vec{\alpha}$	Torque $\vec{\tau}$ $\vec{\tau} = I\,\vec{\alpha}\cdot$ ("mechanic momentum")

state (is kept! inertia!)
(changes, but only continuously)

change the state
(no memory)

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Per-object constant: mass & its distribution (for non point-shaped ones)

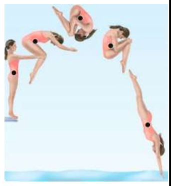
A few quantities associated to each rigid object

- constants: they don't (normally) change
- *input* of the physics dynamic simulation, not output

- Mass:
  - resistance to change of velocity

(distribution of mass:)

- Moment of Inertia:
  - resistance to change of *angular* velocity



- Barycenter:
  - the center of mass

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Mass: notes

- resistance to change of velocity
  - also called *inertial* mass
- also, incidentally:
  ability to attract every other object
  - also called *gravitational* mass
  - happens to be the same
- it's what you measure with a scale
- Unity of measure:
  kg, g, etc…

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Barycenter (of a rigid body): notes

- Aka the center of mass of an object
  - constant: it's a fixed point in *local* space for a rigid body
- Often (but not necessarily) is the origin of the local frame
  - if so, the *position* of a rigid body (the translation of its transform) = the position of its barycenter
- It's the *weighted average* of the positions of the subparts composing an object
  - literally "weighted": with their masses
- In absence of external forces, the object rotates (orbits, spins) around this position.

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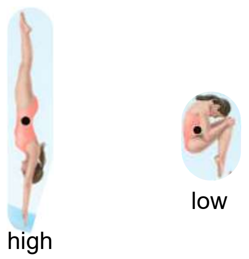
Moment of inertia: notes 1/3

- Resistance to change of angular velocity



- (an object rotates around its barycenter)

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Moment of inertia: notes 2/3

- Scalar moment of inertia
  - Resistance to change of angular velocity
  - Depends on the total mass, and also on its *distribution*
    - the farthest one sub-mass from the axis, the > the resistance
- In 2D: it's a fixed value
  (for a given rigid object)
  - The object always spins around its barycenter

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Moment of inertia: notes 3/3

- In 3D: the rigid objects spins around an axis passing through the barycenter
  - for any possible axis of rotation,
    you have a different *scalar* moment of inertia
  - for a given axis $\hat{a}$ the scalar moment is given by
    $$\hat{a}^{\mathrm{T}}\ \mathrm{M}\ \hat{a}$$
    where 3×3 matrix M is the «(moment of) inertia *matrix*»
    aka the «(moment of) inertia *tensor*»)
  - M can be computed for a given rigid object
    - how: that's beyond this course
    - in practice: use given formulas for common shapes
    - or, sum the contributions for each sub-mass
  - M describes the scalar moment of inertia for any possible axis or rotation

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State of a rigid object in a physical simulation

```
Point position
Rotation orientation
```
current

```
Vector velocity
Vector angular_velocity
```
current rates of change

(current and current rates of change:) updated by physics (dynamics)

```
Scalar mass
Matrix moment_of_inertia
Point barycenter
Scalar drag
...
```
constants — setup at initialization, rarely changes (e.g. by scripts)

(drag:) frictions; see later

*Note: acceleration/forces/torques are **not** part of the state*

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In unity

Adding a "RigidBody" component to a Game Object is to say: *"please let the Physics engine take care of the dynamics of this object"*

```
Vector3 position
Quaternion rotation
```
part of **Transform** component — note: they are the components of the **global** transformation!

```
Vector3 velocity
Vector3 angular_velocity
float mass
Vector3 inertiaTensor
Quaternion inertiaTensorRotation
Vector3 centerOfMass
float drag
bool isKinematic
```
the **RigidBody** component

- velocity — note: speed = velocity.magnitude
- velocity, angular_velocity — per second (not per frame!)
- inertiaTensor, inertiaTensorRotation — **moment of inertia matrix**: the Vector3 = a diagonal matrix D by rotating it $R^T D R$ → the final matrix
- centerOfMass — the barycenter (in local space)
- isKinematic — if true: disable dynamics (but keeps e.g. collisions)

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The case of particles

- For now, we will study a simpler case:
  the dynamics of particles (and its simulation)
- Particle = ideal object shaped like a point,
  with all the mass concentrated in that point
- Particles-only is easier because the following are irrelevant:
  - rotation (orientation): a point does not rotate
  - the center of mass (it's the position of the particle itself);
  - the distribution of mass, i.e. the moment of inertia (there's none);
  - the torques (only forces matter);
  - the angular velocity (there's only linear velocities)
- These things are only relevant for non-point sized (rigid) objects
- The algorithms we are bout to see can be extended to rigid bodies

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State of a particle (point sized obj) in a physical simulation

Point **position**

~~Rotation **orientation**~~

Vector **velocity**

~~Vector **angular_velocity**~~

Scalar **mass**

~~Matrix **moment_of_inertia**~~

~~Point **barycenter**~~

Scalar **drag**

…

not used for point sized objects!

One possibility in a game phys engine is to only simulate point-particles.

Simpler: no rotation needed!

We will see later how to still get rigid bodies back.

For now, we focus on this simpler case.

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Newtonian Dynamics (for particles)

describes the forces
given all the particle positions (and more)

$$\begin{cases} \vec{f}(t) = \text{function}(\,\mathbf{p}(t)\,, \dots) \\ \vec{v}(t) = \dot{\mathbf{p}}(t) \\ \vec{a}(t) = \ddot{\mathbf{p}}(t) = \dfrac{\vec{f}(t)}{m} \\ \dot{\mathbf{p}}(0) = \vec{\mathrm{v}}_0 \\ \mathbf{p}(0) = \mathrm{p}_0 \end{cases}$$

derivative w.r.t. time

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Newtonian Dynamics (an equivalent formulation)

$$\begin{cases} \vec{f}(t) = \text{function}(\,\mathbf{p}(t)\,, \dots) \\ \vec{a}(t) = \dfrac{\vec{f}(t)}{m} \\ \vec{v}(t) = \vec{v}_0 + \displaystyle\int_{t'=0}^{t} \vec{a}(t') \cdot dt' \\ \mathbf{p}(t) = \mathbf{p}_0 + \displaystyle\int_{t'=0}^{t} \vec{v}(t') \cdot dt' \end{cases}$$

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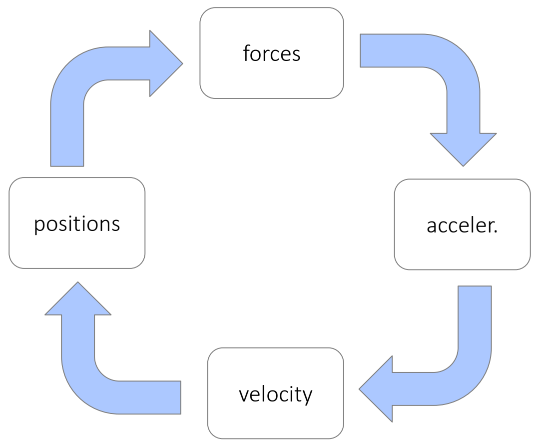
Dynamics (Newtonian)



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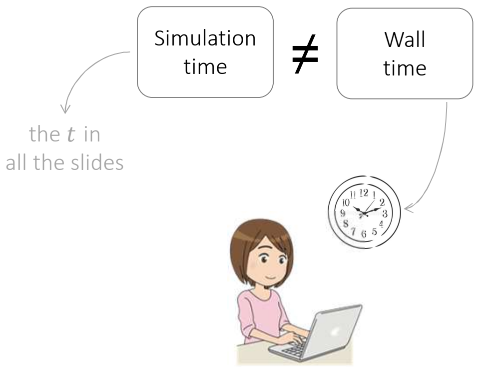
An obvious remark, but



They are just artificially made to flow in sync… usually

- But (e.g.) not when:
  game is paused (t is constant), replays, fast forwards, reverses…

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An obvious remark, but

Simulation time ≠ Wall time

the *t* in all the slides

Occasionally, the difference is spectacularly exploited by clever gameplay designs!

PoP – the sands of times
(Ubisoft, 2003)

The longing
(Studio Seufz, 2020)

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Computing physics evolution

- Analytical solutions:

  state = function(*t*)

  Given force functions (and acc), find the functions (pos, vel,…) in the specified relations:

$$\begin{cases} \vec{f}(t_C) = funz(p(t_C),\ldots) \\ \vec{a}(t_C) = \vec{f}(t_C)/m \\ \vec{v}(t_C) = \vec{v}_0 + \int_0^{t_C} \vec{a}(t)\cdot dt \\ p(t_C) = p_0 + \int_0^{t_C} \vec{v}(t)\cdot dt \end{cases}$$

- Numerical solutions:

  1. $\text{state}_{(t=0)}$ ← init
  2. $\text{state}_{(t+1)}$ ← do_1_step(state_t)
  3. goto 2

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Analytical solutions

that is,
a trajectory:
a position over time

Find the positions as a function $\mathbf{p}(t)$ of time t such that…

a given function

$$\begin{cases} \ddot{\mathrm{p}}(t) = \mathrm{forces}(\,\mathrm{p}(t), \dots)/\mathrm{m} \\ \dot{\mathrm{p}}(0) = \vec{\mathrm{v}}_0 \\ \mathrm{p}(0) = \mathrm{p}_0 \end{cases}$$

derivative w.r.t. time

sometimes,
it's a function of
other things too
(e.g. velocity,
time…).
Harder to solve!

the initial conditions
(for speed and position)

A system of ODE
(Ordinary Differential Equations)

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Analytical solutions

- Difficult to find
  - we need to find a function such that…
- Often, they doesn't even «exist»
  - in a form that we can write using common notations such as polinomials, algebraic functions, exponentials, trigonometry, etc
- But when they exist, they are very convenient to use
  - we can find the position / the velocity for any given *t*
  - we can predict the status of the simulation for any given time
- Examples of systems that admit an analytical solution:
  - systems with a force function is constant w.r.t. positions & velocities (solution: just find its integral, twice)
  - two bodies (no more than two!), subject to reciprocal gravity force
  - a single pendulum, if one accepts an approximation (only good for small oscillations)
- Most other systems don't!

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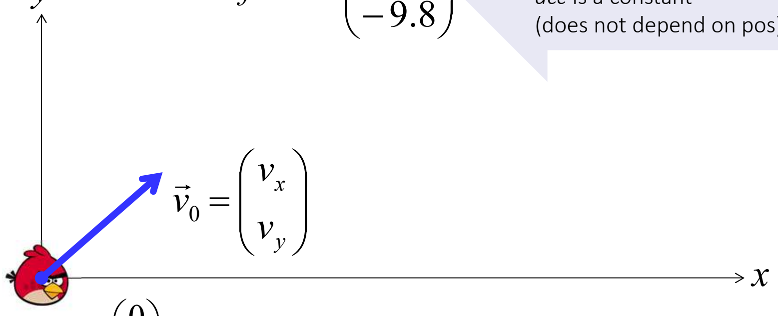
Simple example: analytical solution for...

«ballistic shooting» of a mass, in 2D, ignoring friction...

$$\vec{f} = m \cdot \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

in *this* specific case, *acc* is a constant (does not depend on pos)



$$\vec{v}_0 = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Simple example: analytical solution for...

Solving...

$$\vec{f}(t_C) = fun(p(t_C),...)$$
$$\vec{a}(t_C) = \vec{f}(t_C)/m$$
$$\vec{v}(t_C) = \vec{v}_0 + \int_0^{t_C} \vec{a}(t) \cdot dt$$
$$p(t_C) = p_0 + \int_0^{t_C} \vec{v}(t) \cdot dt$$

$$\vec{f}(t_C) = m \cdot \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\vec{a}(t_C) = \vec{f}(t_C)/m = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\vec{v}(t_C) = \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \int_0^{t_C} \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \cdot dt = \begin{pmatrix} v_x \\ v_y - 9.8 \cdot t_C \end{pmatrix}$$

$$p(t_C) = p_0 + \int_0^{t_C} \vec{v}(t) \cdot dt = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^{t_C} \begin{pmatrix} v_x \\ v_y - 9.8 \cdot t \end{pmatrix} \cdot dt = \begin{pmatrix} v_x \cdot t_C \\ v_y \cdot t_C - 9.8/2 \cdot t_C{}^2 \end{pmatrix}$$

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Simple example: analytical solution for…

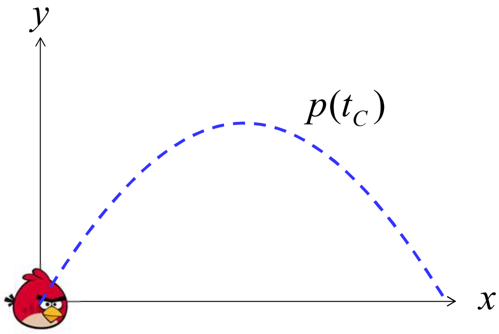
Final result:

$$\vec{f}(t_C) = m \cdot \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\vec{a}(t_C) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\vec{v}(t_C) = \begin{pmatrix} v_x \\ v_y - 9.8 \cdot t_C \end{pmatrix}$$

$$p(t_C) = \begin{pmatrix} v_x \cdot t_C \\ v_y \cdot t_C - 9.8/2 \cdot t_C{}^2 \end{pmatrix}$$



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Numerical integration

$$\vec{f}(t_C) = \text{function}(p(t_C), \dots)$$

$$\vec{a}(t_C) = \vec{f}(t_C)/m$$

$$\vec{v}(t_C) = \vec{v}_0 + \int_0^{t_C} \vec{a}(t) \cdot dt$$

$$p(t_C) = p_0 + \int_0^{t_C} \vec{v}(t) \cdot dt$$

It’s our way to solve the ODE

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Numerical integration

- A numerical integrator computes the integral as summed area of small rectangles
  - For a physics engine, this means just updating velocity and positions at each physics step
- A crucial parameter is the width of the rectangles i.e. *dt* = the duration of the physics step (in virtual time)
  - If physics system perform *N* steps per second: *dt* = 1.0 sec / N
  - *N* is not necessarily same rendering frame rate e.g.: rendering 30 FPS but physics: 60 steps per seconds
  - *dt* is not necessarily constant during the simulation (but in most system, it is)

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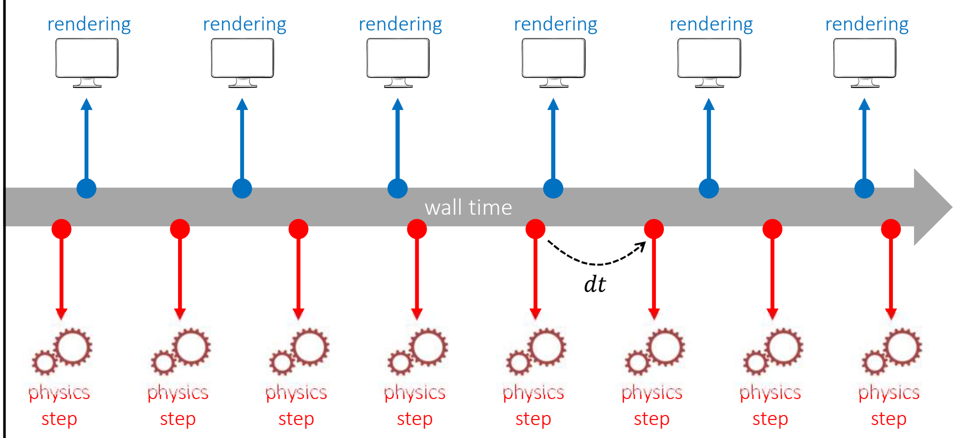
Rendering *Frames-per-Seconds* (FPS) vs Physics *Steps-per-Seconds*



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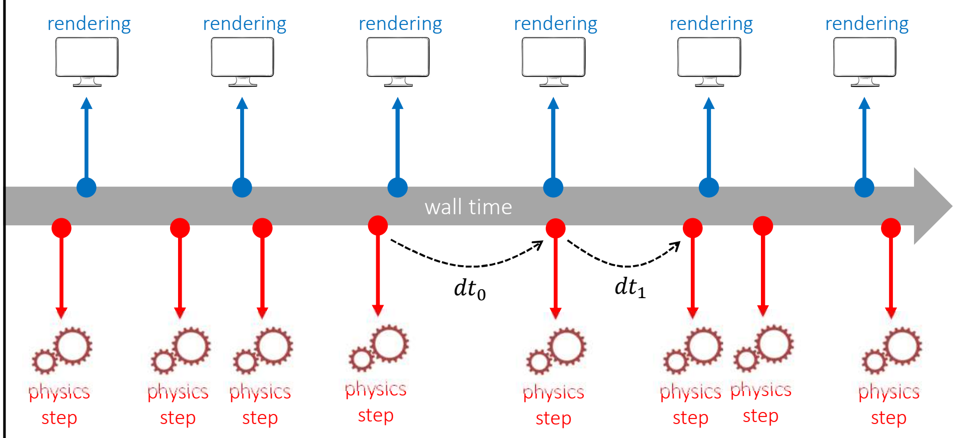
In principle, timestep duration can be different in each step



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Numerical methods: features

- How efficient / expensive
  - must be at least soft real-time
    - (if from time to time computation delayed to next frame, ok)
- How accurate
  - must be at least plausible
    - (if stays plausible, differences from reality are acceptable)
- How robust
  - rare completely wrong results
    - (and never crash)
- How general
  - Which phenomena / constraints / object types is it able to recreate?
  - requirements depend on the context (ex: gameplay)

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Euler integration methods

For each step:

$$\vec{f} = fun(\mathrm{p}, \dots)$$

(1) Evaluate the force
on each particle
as a function of positions
(of this and/or other particles)
and any other things needed things too

$$\vec{a} = \vec{f}/m$$

(2) acceleration
of each particle given by:
total force acting on it divided by its mass

$$p = p_0 + \int \vec{v} \cdot dt$$

(3) Update position with velocity

$$\vec{v} = \vec{v}_0 + \int \vec{a} \cdot dt$$

(4) Update velocity with acceleration

green = state variables
blue = temp variables

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Euler integration methods

init

$$\mathbf{p} \leftarrow \cdots$$
$$\vec{\mathrm{v}} \leftarrow \cdots$$

one step

$$\vec{\mathrm{f}} \leftarrow fun(\mathbf{p}, \dots)$$
$$\vec{\mathrm{a}} \leftarrow \vec{\mathrm{f}}/m$$
$$\mathbf{p} \leftarrow \mathbf{p} + \vec{\mathrm{v}}\ dt$$
$$\vec{\mathrm{v}} \leftarrow \vec{\mathrm{v}} + \vec{\mathrm{a}}\ dt$$

$$t = t + dt$$

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Forward Euler *pseudo code*

```
Vec3 position = …
Vec3 velocity = …

void initState(){
   position = …
   velocity = …
}

void physicStep( float dt )
{
   Vec3 acceleration = compute_force( position ) / mass;
   position += velocity     * dt;
   velocity += acceleration * dt;
}

void main(){
  initState();
  while (1) do physicStep( 1.0 / FPS );
}
```

Equivalent to…

$$\vec{f}_i = function(p_i, \dots)$$
$$\vec{a}_i = \vec{f}/m$$
$$\vec{v}_{i+1} = \vec{v}_i + \vec{a}_i \cdot dt$$
$$p_{i+1} = p_i + \vec{v}_i \cdot dt$$

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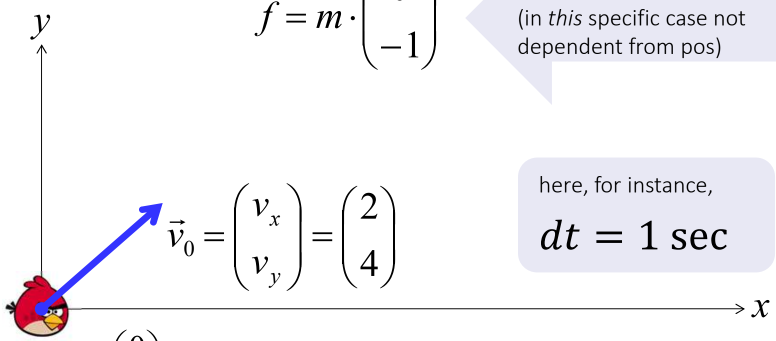
Simple example: numerical solution

Same phenomena of previous example

$$\vec{f} = m \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

constant (in *this* specific case not dependent from pos)



$$\vec{v}_0 = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

here, for instance, $dt = 1$ sec

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Simple example: numerical solution (with dt=1 sec)

init

Time:	0	1	2	3	4	5	6	7	...
vel:	(2,3)	(2,2)	(2,1)	(2,0)	(2,-1)	(2,-2)	(2,-3)	(2,-4)	...
pos:	(0,0)	(2,3)	(4,5)	(6,6)	(8,6)	(10,5)	(12,3)	(14,0)	...

step step step step step step step step

$$\vec{f} = m \cdot \binom{0}{-1}$$

$$\vec{a} = \vec{f} / m$$

$$\vec{v} = \vec{v} + \vec{a} \cdot dt$$

$$p = p + \vec{v} \cdot dt$$

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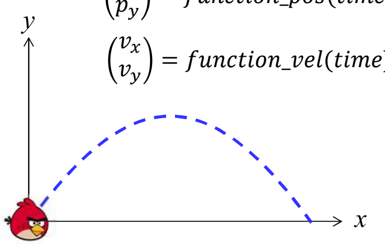
Physics evolution computation

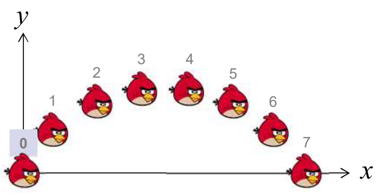
- Analytical solutions:

$$\binom{p_x}{p_y} = function\_pos(time)$$

$$\binom{v_x}{v_y} = function\_vel(time)$$



- Numerical solutions:



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Physics evolution computation

- Analytical solutions:
  - Super efficient!
    - Close form solution
  - Accurate
  - Only simple systems
  - Formulas found case by case (often they don't even exist)
  - NOT USED (but, for instance, useful to to make predictions for, e.g. A.I.)
- Numerical solutions:
  - Expensive (iterative)
    - but *interactive*
  - Integration errors
  - Flexible
  - Generic
  - USED FOR DYNAMICS

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Integration errors

- A numerical integrator only approximates the actual value of the integrals
- The discrepancy (simulation errors) accumulates with virtual time during all the simulation
- How much error is accumulated?
- It depends on dt
  - smaller $dt \Rightarrow$ smaller error (simulation is more accurate)

  but, clearly
  - smaller $dt \Rightarrow$ more steps are needed (for simulate the same virtual time) $\Rightarrow$ simulation is more computationally expensive, but smaller errors,

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Order of convergence

- How much does the total error decrease as *dt* decreases?
  - That's called the Order of the simulation
  - 1st order: the total error can be as large as O(dt^1)
    - "if the number of physics steps doubles (physical computation effort doubles) *dt* becomes halves and errors can be expected to halve"
    - The error introduced by each single step is O(dt^2),
  - The Euler seen is 1st order
    - This is not too good, we want better
    - Note: The error is usually not that bad as linear with *dt*, but they *can* be

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The integration step dt of any numerical methods (summary)

dt : delta of virtual time from last step

- the "temporal resolution" of the simulation!

number of physics steps per sec, or «physics FPS»

- if large: more efficiency
  - fewer steps to simulate same amount of virtual time
- if small: more accuracy
  - especially with strong forces and/or high velocities
- Common values: 1 sec / 60 … 1 sec / 30
  - i.e. a step simulates around 16 … 32 msec. of virtual time
  - note: it's not necessarily the same refresh rate of rendering (FPS of rendering ≠ FPS of physics. Rendering can be *less*!)
  - note: dt is not necessarily the same in all physics steps (need more accuracy *now*? Decrease *dt*

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Effect of integration errors of System Energy

- Because of integration errors:
  simulated solutions ≠ "real" solutions
- In a real system, the total energy can never increase
  - typically, it *decreases* over time, due to dissipations
  - that is, attrition turns *dynamic energy* into *heat*
- Therefore, a particularly nasty integration error is when the total energy of the system *increases* over time
  - e.g.: a pendulum swings wider and wider
- Particularly bad because:
  - compromises stability
    (velocity = big, displacements = crazy, error = crazy)
  - compromises plausibility
    (we can *see* it's wrong)
- A simple way to avoid this:
  make sure the simulation always includes attritions
  - makes simulation more stable + robust

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Other numerical integrators ("numerical ways to compute integrals")

- Some commonly used alternatives (among MANY!):
  - "Forward" Euler method (the one seen so far)
  - Symplectic Euler method
  - Leapfrog method (next lecture)
  - Verlet method (next lecture)
- These are just variants of each other – let's see them!
  - From the code point of view, no big change
  - They can differ in accuracy / behavior
  - They can have different "orders of accuracy"
  - Note: a more accurate method is also more efficient
    (larger dt are possible, so fewer steps are necessary)

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Forward Euler Method: limitations

- efficiency / accuracy: not too good
  - error accumulated over time = linear in dt
  - it's only a "first order" method
  - Doubles the steps = halve the *dt* , only halves the errors (can be better, but no guarantees)
- scarce stability for large *dt*
- minor problem: no reversibility, *even in theory*
  - real Newtonian Physics is reversible: flip all velocities and forces ⇒ go backward in time.
  - In our simulation (with Euler): this doesn't work exactly
  - Ability to go reverse a simulation would be useful in games! E.g. replays in a soccer game ?
  - Pro tip: basically, reverse time direction never done like this To go backward in time accurately, store states

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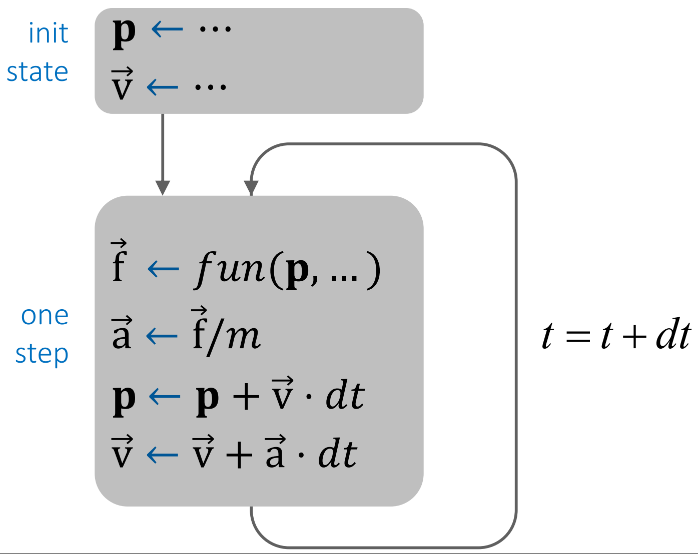
Forward Euler



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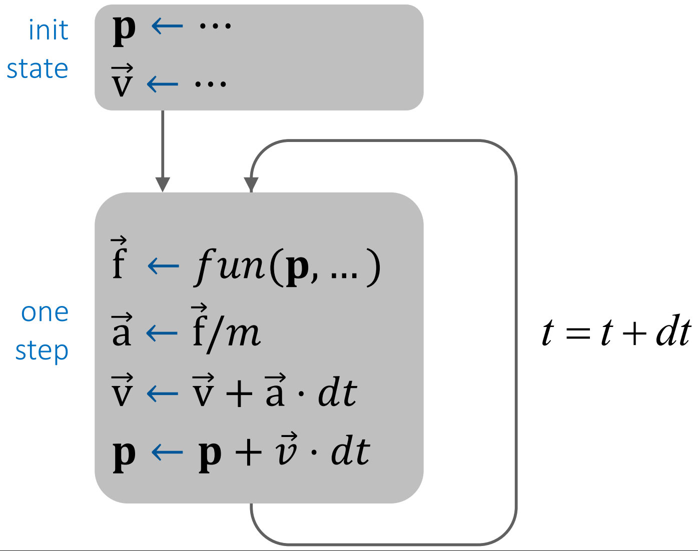
Symplectic Euler



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Forward Euler *pseudo code*

Equivalent to...

$$\vec{f}_i \leftarrow function(p_i, \ldots)$$
$$\vec{a}_i \leftarrow \vec{f}/m$$
$$\vec{v}_{i+1} \leftarrow \vec{v}_i + \vec{a}_i \cdot dt$$
$$p_{i+1} \leftarrow p_i + \vec{v}_i \cdot dt$$

```
Vec3 position = …
Vec3 velocity = …

void initState(){
   position = …
   velocity = …
}

void physicStep( float dt )
{
   Vec3 acceleration = compute_force( position ) / mass;
   position += velocity     * dt;
   velocity += acceleration * dt;
}

void main(){
  initState();
  while (1) do physicStep( 1.0 / FPS );
}
```

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Symplectic Euler *pseudo code* (aka semi-implicit Euler)

```
Vec3 position = …
Vec3 velocity = …

void initState(){
   position = …
   velocity = …
}

void physicStep( float dt )
{
   Vec3 acceleration = compute_force( position ) / mass;
   velocity += acceleration * dt;
   position +=     velocity * dt;
}

void main(){
  initState();
  while (1) do physicStep( 1.0 / FPS );
}
```

just flip the order

Equivalent to…

$$\vec{f}_i \leftarrow function(p_i, \ldots)$$
$$\vec{a}_i \leftarrow \vec{f}/m$$
$$\vec{v}_{i+1} \leftarrow \vec{v}_i + \vec{a}_i \cdot dt$$
$$p_{i+1} \leftarrow p_i + \vec{v}_{\boldsymbol{i+1}} \cdot dt$$

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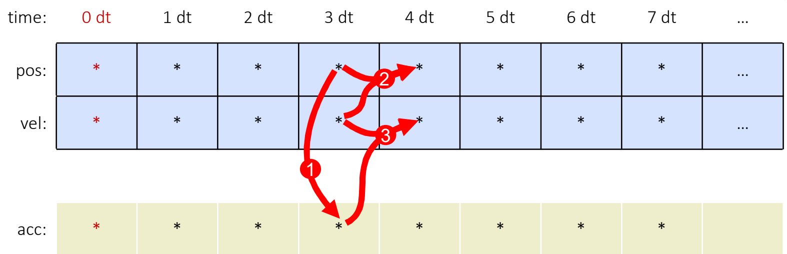
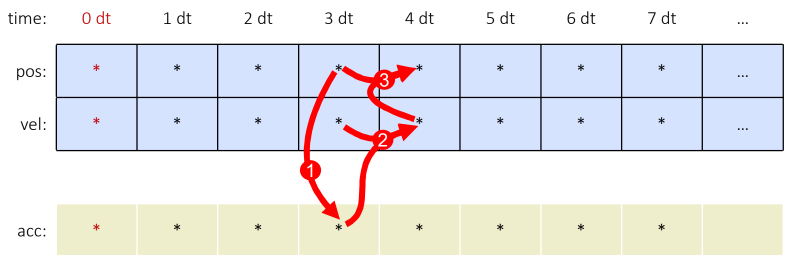
Forward Euler:



Symplectic Euler:



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Forward Euler VS Symplectic Euler (warning: over-simplifications)

- From the code point of view, they are very similar
- The semantics changes:
  - in Symplectic Euler
    the position altered using *next frame* velocity
  - (it's "wrong", in a sense – but tends to work better)
- Similar properties, but better in practice
  - Same order of convergence (still just 1 ☹)
  - On average,
    Symplectic tends to be more stable and accurate

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Forces: examples

$$\dots$$
$$\vec{f} = \text{function}(\mathbf{p}, \dots)$$
$$\dots$$

Primarily, a function of the positions

But not always, and sometimes not only of positions (also: velocities? Global time?)

- Gravity
  - Constant · *m*, near the surface of a planet
  - Function of positions in a space simulation
- Wind pressure
  - Depends on the area exposed in the wind direction
- Electrical / magnetic forces
- Buoyancy *(ita: forza di Archimede)*
  - Depends on the weight of the submerged volume
- Mechanical springs
  - simple model: Hooke's law – see later
- Shock waves (explosions)
- Fake / "Magic" control forces
  - added for controlling the evolution of the system,
    not physically justified

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Example of forces: gravitational force on a planet surface

- Given a particle with (gravitational) mass m

some global constant dependent on… the planet

$$\overrightarrow{f_g} = g\, m \;\; \hat{\mathrm{d}}_{\mathrm{DOWN}}$$

force magnitude (positive scalar)

force direction (versor)

Notes:
- it does not depend on position, (assuming that the distance from the center of the planet doesn't change much)
- linear with (gravitational) mass
- will produce a constant acceleration (regardless of mass!) when divided by (inertial) mass m

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Example of forces: gravitational force (in open space)

- A particle A in pos $\mathbf{p}_a$ with (gravitational) mass m_a is attracted by a particle B in pos $\mathbf{p}_b$ with (gravitational) mass m_b by a force

some global constant dependent on… the universe

$$\overrightarrow{f_a} = \frac{\mathbf{p}_b - \mathbf{p}_a}{||\mathbf{p}_b - \mathbf{p}_a||} \; \frac{g\, m_a\, m_b}{||\mathbf{p}_b - \mathbf{p}_a||^2} =$$

force direction, from A to B (versor)

force magnitude (positive scalar)

$$= (\mathbf{p}_b - \mathbf{p}_a) \frac{g\, m_a\, m_b}{||\mathbf{p}_b - \mathbf{p}_a||^3}$$

note: and B is also attracted by A, by exactly the opposite force $\overrightarrow{f_b} = -\overrightarrow{f_a}$

as seen in Space Wars, 1962

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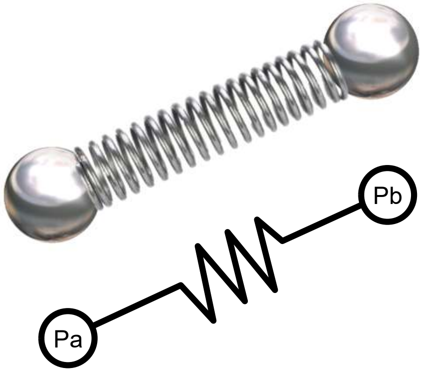
Forces: Springs



Hooke's law:

$$\vec{f_a} = k(\|\mathbf{p}_b - \mathbf{p}_a\| - \ell)\underbrace{\frac{\mathbf{p}_b - \mathbf{p}_a}{\|\mathbf{p}_b - \mathbf{p}_a\|}}_{\text{force direction (versor)}}$$

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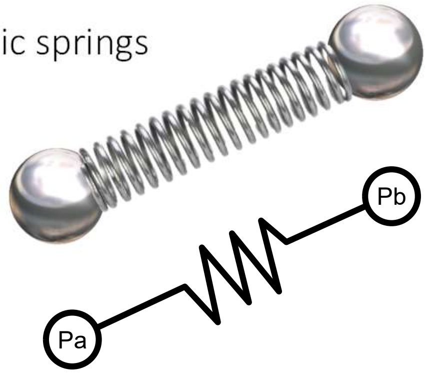
Forces: Springs (Hooke's law)

- Simplified model for elastic springs
- One spring connects two particles in $\mathbf{p}_a$ and $\mathbf{p}_b$
- Characterized by:
  1. Rest length ℓ
  2. Stiffness k
- Spring force: counteracts expansion and compression



$$\vec{f_a} = k(\|\mathbf{p}_b - \mathbf{p}_a\| - \ell)\frac{\mathbf{p}_b - \mathbf{p}_a}{\|\mathbf{p}_b - \mathbf{p}_a\|}$$

$$\vec{f_b} = -\vec{f_a}$$

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Forces: Springs (Hooke's law)

$$\vec{f_a} = k\,\underbrace{(\overbrace{\|\mathbf{p}_b - \mathbf{p}_a\| - \ell}^{\text{elongation / compression}})}_{\text{force magnitude (scalar) (positive or negative)}}\;\underbrace{\frac{\mathbf{p}_b - \mathbf{p}_a}{\|\mathbf{p}_b - \mathbf{p}_a\|}}_{\text{force direction (versor)}}$$

⇐ force to be applied to particle a

$$\vec{f_b} = - \vec{f_a}$$

⇐ force to be applied to particle b

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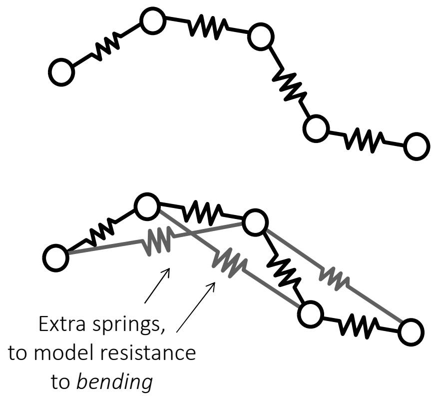
Mass and Spring systems

- Useful for deformable objects
- for instance: elasitic ropes (or hairs)



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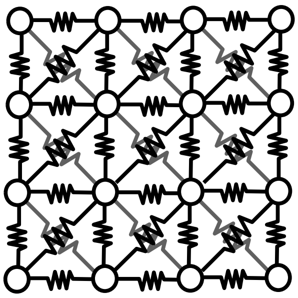
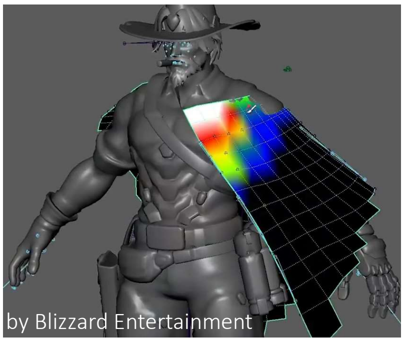
Mass and Spring systems

- For instance: cloth





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Mass and Spring systems can model…

- Elastic deformable objects (aka "soft bodies")
  - Elastic = go back to original shape
  - Easily modelled as compositions of (ideal) springs.
- Plastic deformable objects? (yes, but not easy)
  - Plastic = assume deformed pose permanently
  - Dynamically change rest-length L in response to large compression/stretching, in certain conditions (not easy)
- Rigid bodies / inextensible ropes ? (no they can't)
  - Increase spring stiffness? $k \to \infty$
  - Makes sense, physically, but…
  - Large $k \Rightarrow$ large f $\Rightarrow$ instability $\Rightarrow$ unfeasibly small dt needed
  - Doesn't work. How, then? see later

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