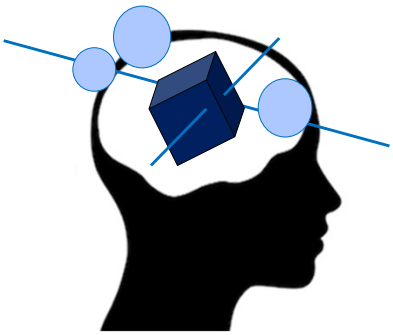
3D VideoGames - UniMi

Points, Vectors & Versors algebra



Marco Tarini

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Course Plan

lec. 1: **Introduction**
lec. 2: **Mathematics** for 3D Games
lec. 3: **Scene Graph**
lec. 4: Game **3D Physics** +
lec. 5: Game **Particle Systems**
lec. 6: Game **3D Models**
lec. 7: Game **Materials**
lec. 8: Game **Textures**
lec. 9: Game **3D Animations**
lec. 10: **Audio** for 3D Games
lec. 11: **Networking** for 3D Games
lec. 12: **Interactive Agents** for 3D Games
lec. 13: **Rendering Techniques** for 3D Games

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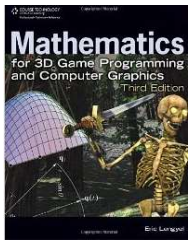
Suggested reading



Mathematics for 3D Game Progr. and C.G. (3rd ed)
Eric Lengyel
Chapters 2, 3, 4

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Scalars, Points, Vectors, Versors & associated oprerations

They are the basic building blocks

- In the computation, for all videogame tasks
  - rendering
  - physic symulations
  - AI
  - 3D sound
  - …
- in the data structures of all 3D Assets
  - meshes, animations, etc

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Point, Vectors, Versors

	represents:	example:	imagine it as...
Point	A position A location	Where a character is The center of a sphere	a small floating dot :-D
Vector	A displacement The difference between 2 points. The vector that connects them.	The velocity of a thrown knife The gravity acceleration How to reach the head of a character from its neck	a small arrow :-D *(length is relevant)*
Versor aka unit vector (as length = 1) aka normal aka direction aka normalized vector	A direction A facing	The view direction of a character The facing of a plane in 3D (i.e. its "normal") The direction of a line, or a ray A rotation axis	the same :-D *(its length is irrelevant)*

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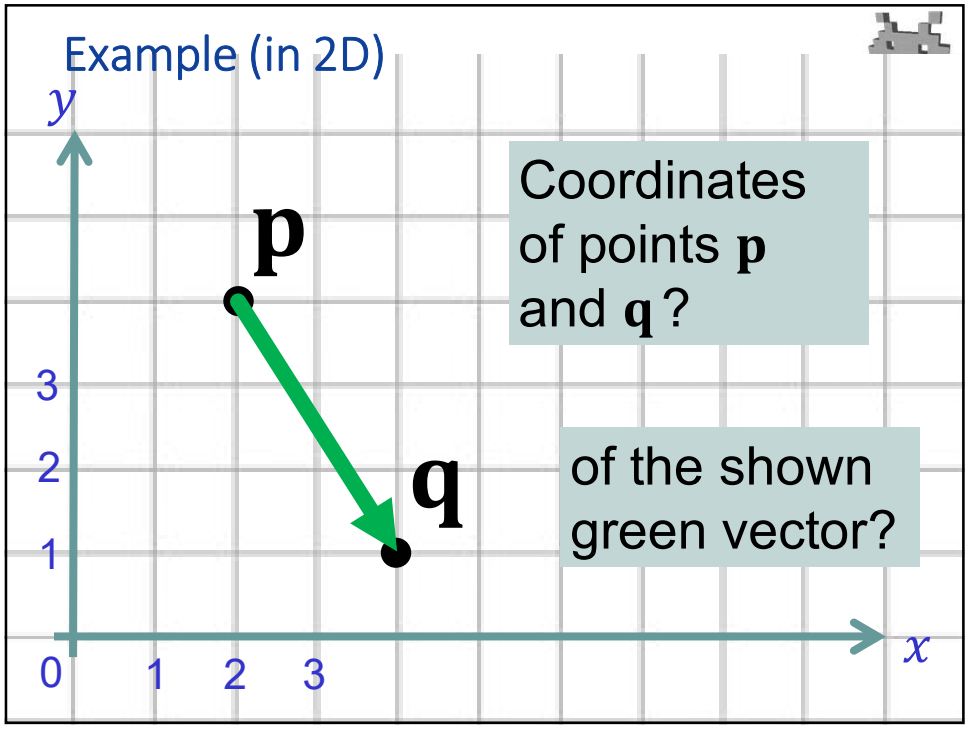
Example (in 2D)



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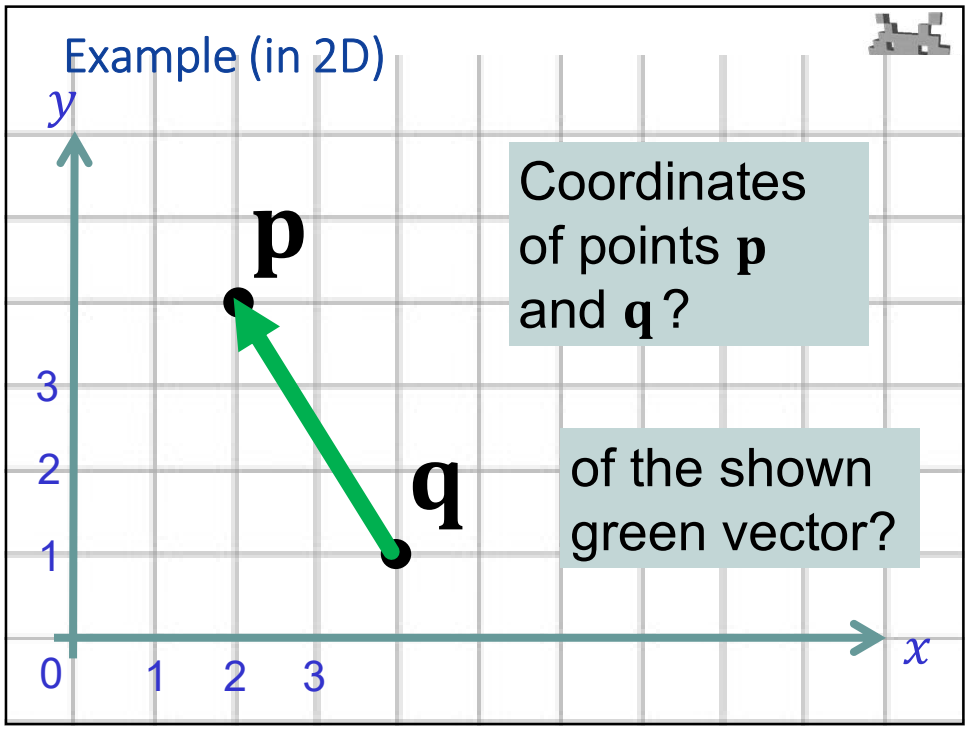
Example (in 2D)
y
x
0
1
2
3
p
q
Coordinates of points p and q ?
of the shown green vector?

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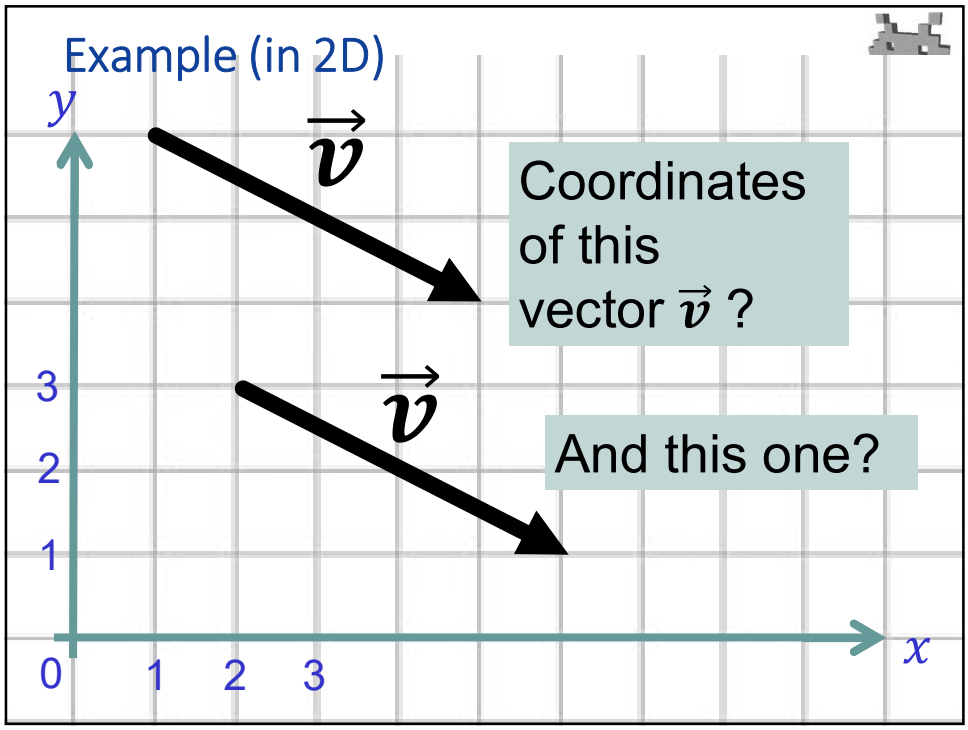
Example (in 2D)
y
x
0
1
2
3
$\vec{v}$
$\vec{v}$
Coordinates of this vector $\vec{v}$?
And this one?

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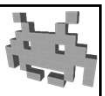
Points, Vectors, Versors: internal representation



Thinking in terms of programming languages,
it's helpful to imagine them as three *distinct* types of object

coords

deltas

no direct write-access
(must ensure unitarity)

```
class Point {
  // fields:
  float x,y,z;

  // methods:
  …
}
```

```
class Vector {
  // fields:
  float dx,dy,dz;

  // methods:
  …
}
```

```
class Versor {
  // fields:
  const float dx,dy,dz;

  // methods:
  …
}
```

- Each supporting their own *distinct* operations & methods
- For now. In practice, many (all?) existing 3D libs choose to use the same class for the three of them (see later)

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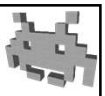
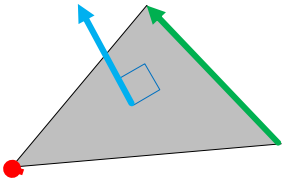
Points, Vectors, Versors …on a 3D floating tirangle



Examples of…

- point:
  - one vertex of the triangle
- vector:
  - one side of the triangle
- versor:
  - the «normal» of the triangle



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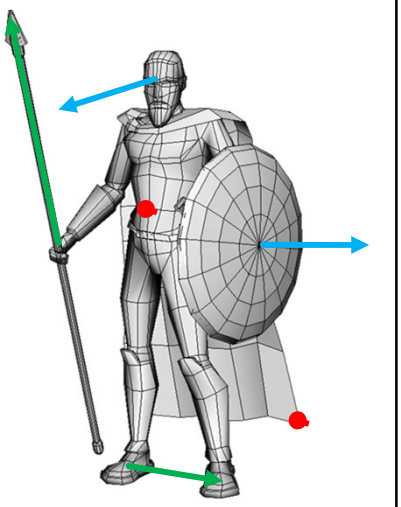
Points, Vectors, Versors …in a character model



Examples of…

- points:
  - the pos of the navel
  - the pos of lewer-left tip of the hood
- vectors:
  - the vector connecting the L foot to the R foot
  - the vector from the hand to the tip of the lance
- versors:
  - the gaze direction
  - the facing of the shield

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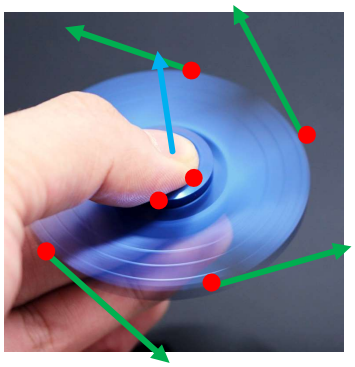
Points, Vectors, Versors …in a spinner



Examples of…

- points:
  - points of contact between finger-spinner
- vectors:
  - linear velocities of these four points
- versors:
  - rotation axis (direction of)

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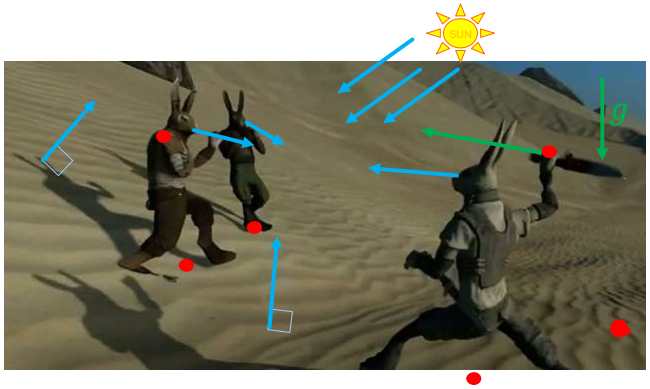
Points, Vectors, Versors
...in this screenshot



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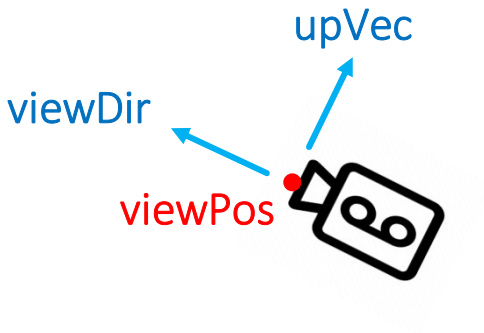
Stuff = Points + Vectors + Versors



Description of a camera
(its so called «extrinsic parameteres»)

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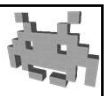
Stuff = Points + Vectors + Versors



description of
a (directional) sound emitter

description of
a (directional) microphone

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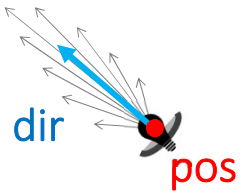
Stuff = Points + Vectors + Versors



description of a spotlight

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The *algebra* of points, vectors, versors (and scalars)

- make sure you understand each of operation in 3 different ways:

  - intuitive / spatial: what does it do conceptually / visually in 3D

  - operational: how to compute the result, starting from
    (a) the coordinates of the operand(s)
    (b) when appropriate, also
    the angle between the operands, their lengths, etc

  - syntactic: how to write them down
    (a) on paper (mathematical notation)
    (b) in a programming language (Unity C# lib, Unreal C++ lib, GLSL…)

- also, familiarize how to manipulate equations, i.e. rules such as

(a) commutativity? associativity? (of each operation)
(b) distributivity? (between pairs of operations)
(c) inverse operation? identity element? absorbing element?

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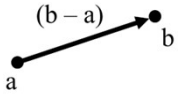
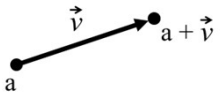
Point and vector algebra (summary 1/7)

- Difference:
  point – point = vector



- Addition:
  point + vector = point



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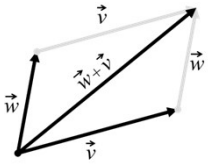
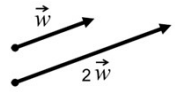
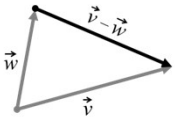
Point and vector algebra (summary 2/7)



- Linear operations for vectors
  - addition (vector + vector = vector)
  - product with a scalar (scaling)
    (vector * scalar = vector)
  - therefore: interpolation
    $\text{mix}(\overrightarrow{v_0}, \overrightarrow{v_1}, t) = (1-t)\,\overrightarrow{v_0} + t\,\overrightarrow{v_1}$
  - therefore: opposite (flip verse)
    (how to: multiply by – 1)
  - therefore: difference
    (vector – vector = vector)





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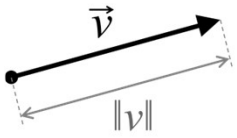
Point and vector algebra (summary 3/7)



- Norm (for vectors)
  - aka length / magnitude /
    Euclidean norm / 2-norm
  - distance between points:
    length of vector (**a** – **b**) = distance between **a** and **b**
  - Rules: triangle inequality:

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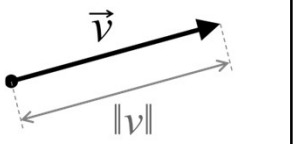
Point and vector algebra (summary 4/7)



- Normalization
  - input: a vector. Result: a versor
  - how to: scale the vector by (1.0 / length)

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Point and vector algebra (summary 5/7)

- Interpolate between pairs of <*something*> :
  - mix(point , point , *t*) → point
  - mix(vector , vector , *t*) → vector
  - mix(versor , versor , *t*) → versor
- *t* is a scalar «weight»
  - *t* = 0 → pick the first one
  - *t* = 1 → pick the second one
  - *t* ∈ (0,1) → get something in between, for example:
  - *t* = 0.5 → just average the two
  - *t* = 0.1 → use almost the first, with just a bit of the second in it
  - *t* < 0 or t > 1 → extrapolate
- Terminology: (in libraries, game engines…)
  - interpolate = mix = blend = lerp

a proper interpolation

specifically linear

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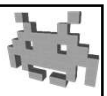
Interpolation in general terms - notes



- Very used in Computer Graphics (e.g., rendering, animation)
- Terminology:
  - $a\,\mathbf{x} + b\,\mathbf{y}$: a linear combination of **x** and **y**
  - if $a+b=1$ and $a,b \in [0,1]$: a (linear) interpolation of **x** and **y**
  - if $a+b=1$ but $a,b \notin [0,1]$: a (linear) extrapolation of **x** and **y**
  - a , b : the used weights
  - $a + b = 1$: weights are a partition of unity
- Generalizes to > 2 objects ($a\,\mathbf{x} + b\,\mathbf{y} + c\,\mathbf{z}$)
- When interpolating 2 objects, we can just specify one weight t.
  - The other is given by difference. $a = t$, $b = 1-t$
  - The interpolation is often written in programming languages as `mix( x, y, t )` (or similar ways). Rember: t=0 → pick the first
- It's a general concept! All sorts of things can be interpolated
  - Intuition: interpolation = a mix between objects (e.g.: colors, statuses…)
  - Let's analyze case of Points, Vectors, Versors

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How to interpolate between…

but easily generalizes to > 2

linear interpolation

- …two vectors $\mathbf{v}_0$ and $\mathbf{v}_1$:

$$(1 - t)\,\mathbf{v}_0 + (t)\,\mathbf{v}_1$$

only legal operations, with an easily defined geometric meaning (to-do: check)

- …two points $\mathbf{p}_0$ and $\mathbf{p}_1$:

$$\mathbf{p}_0 + t\,(\mathbf{p}_1 - \mathbf{p}_0)$$

which you may also want to write as:

$$(1 - t)\,\mathbf{p}_0 + (t)\,\mathbf{p}_1$$

summing… two points ??

scaling… a point ??

we usually don't need any such operation, but it's equivalent, mathematically.

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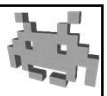
How to interpolate between…



But easily generalizes to > 2

Linear interpolation

- …two vectors $\mathbf{v}_0$ and $\mathbf{v}_1$:

$$(1 - t)\,\mathbf{v}_0 + (t)\,\mathbf{v}_1$$

- …two points $\mathbf{p}_0$ and $\mathbf{p}_1$:

$$\mathbf{p}_0 + t\,(\mathbf{p}_1 - \mathbf{p}_0)$$

- …two versors $\mathbf{d}_0$ and $\mathbf{d}_1$:

$$(1 - t)\,\mathbf{d}_0 + (t)\,\mathbf{d}_1$$

then renormalize the result (it's no longer unitary).

Or, use "spherical interpolation" (aka "slerp")…

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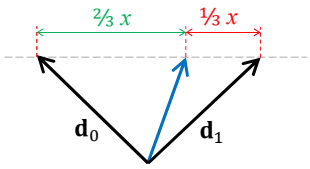
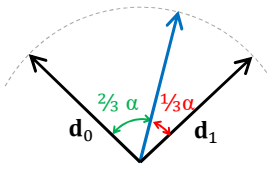
LERP vs SLERP (of versors)

Linear interpolation:



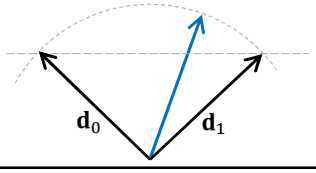
$\mathbf{d} = \mathrm{lerp}(\mathbf{d}_0, \mathbf{d}_1, ⅔)$

Spherical interpolation:



$\mathbf{d} = \mathrm{slerp}(\mathbf{d}_0, \mathbf{d}_1, ⅔)$

then, renormalize:



Not the same result!

- But, close enough
- Even closer when:
  $\mathbf{d}_0$, $\mathbf{d}_1$ similar OR t close to ½
- Is it worth the extra computation cost? 😉

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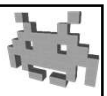
The formulas



- LERP + normalization:

$$(1 - t)\,\mathbf{d_0} + t\,\mathbf{d_1}$$
then re-normalize

} aka "NLERP"

- or SLERP:

angle between $\mathbf{d}_0$ and $\mathbf{d}_1$

$$\frac{\sin((1-t)\,\alpha)}{\sin(\alpha)}\mathbf{d_0} + \frac{\sin(t\,\alpha)}{\sin(\alpha)}\mathbf{d_1}$$

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SLERP: notes

- Applicable to any versor (unit vector) including 2D, 3D, and quaternions (see later)
- SLERP can even be used on general vectors:
  - Compute magnitudes of vectors
  - Compute directions of vectors (divide by magnitude, i.e., normalize)
  - new direction = SLERP of the directions (unit vectors)
  - new magnitude = LERP of the magnitudes (scalars)
  - multiply new dir with new mag to get the final result

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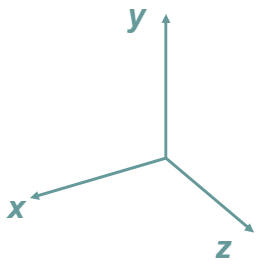
recap: Vector Base



- Axes: set of *n* lin. ind. vectors (***x***,***y***,***z***)
- Any vector ***v*** can be expressed in exactly 1 way as a linar combination of these vectors
- The weights are the coord of ***v*** in that base

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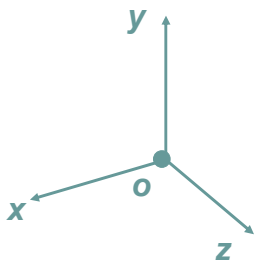
recap: Reference Frame (or Space)



- *n* axes (vectors) (vector base)
  +
  1 origin (point)
- Any vector ***v*** :
  one linear comb. of the axes
- Any point ***p*** :
  origin + one linear comb. of axes

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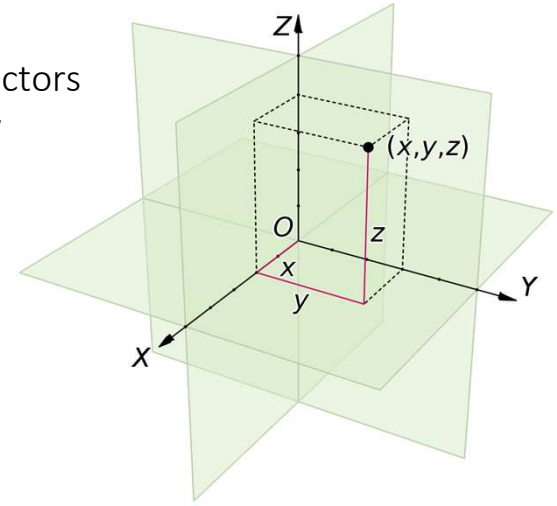
Recap: Orthonormal Frames Or Cartesian Frame

- Axes are unit vectors and reciprocally orthogonal



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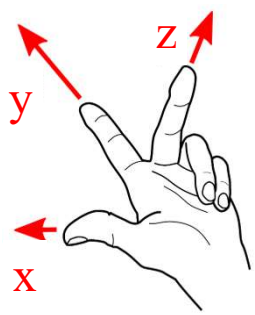
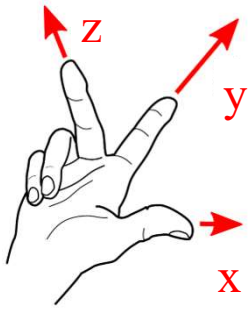
Recap: Handed-ness of a (Cartesian) frame

- They can be right- or left-handed



$x \times y = z$



$x \times y = z$
regardless!

Use the same hand to *imagine* a cross product

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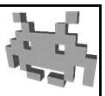
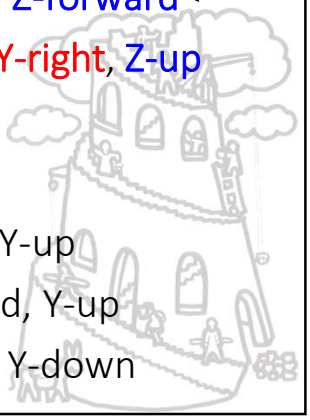
Semantics associated to X,Y,Z axes: still no standards for 3D games



personal opinion:
the most standard one,
among
3D modellers too

- Unity: left-handed: X-right, Y-up, Z-forward
- Unreal: left-handed: X-forward, Y-right, Z-up
- 3ds-Max: right-handed, Z-up
- Blender: left-handed, Z-up
- most VR systems: right-handed, Y-up
- OpenGL: (clip space) right-handed, Y-up
- DirectX: (clip space) left-handed, Y-down



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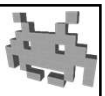
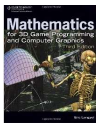
Point and vector algebra
Products: additional reading



To be continued!

Products between vectors and/or versors

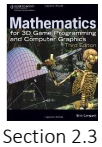


Section 2.2

- Dot product (or inner product)
  - Output: a scalar

NEXT LECTURE!

- Cross product (or vector product)
  - Output: a vector



Section 2.3

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