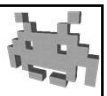
Course Plan



lec. 1: Introduction
lec. 2: Mathematics for 3D Games
lec. 3: Scene Graph
lec. 4: Game 3D Physics
lec. 5: Game Particle Systems
lec. 6: Game 3D Models
lec. 7: Game Textures
lec. 8: Game Materials
lec. 9: Game 3D Animations
lec. 10: 3D Audio for 3D Games
lec. 11: Networking for 3D Games
lec. 12: Interactive Agents for 3D Games
lec. 13: Rendering Techniques for 3D Games

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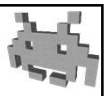
Attrition (or friction) forces



- *Isotropic* friction force : $\vec{f} = -k\,\vec{v}$
  - a force that opposes motion, dissipating energy
  - isotropic = magnitude independent on the direction of motion
  - direction of force: always opposite of current velocity direction
  - Magnitude of force: proportional to the speed = $||\vec{v}||$
  - note: this force depends on velocity, not position(s).
  - models the resistence of the medium where the motion happens (air, water, thin space…)
  - the denser the medium, the largest k, stronger the force (water >> air >> thin space)
- Planar friction forces:
  - A force that happens when things slide against each other
  - Always parallel to the contact plane (orthogonal to the normal)
  - Part of the collision response (see next topic!)

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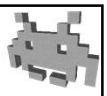
Attrition (or friction) forces: 💡simulate them with *velocity damping* !



- A simpler way to simulate (isotropic) friction: "velocity damping" (or "viscous damping")
  - simply reduce all velocity vectors by a fixed proportion
  - for example: scale velocity down by 2% per second ("drag factor" = 0.02 / sec)
    (that is, scale velocity vectors by a factor 0.98)
  - Why it makes sense:
    Higher speed = more attrition = more loss of speed.
    So, attrition = a "fixed tax" (in %) on velocity.
- For planar friction:
  - Split velocity into parallel / orthogonal parts
  - Apply different Drag factors to each parts

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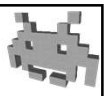
Velocity Damping: how to (as an example)



- Objective: "reduce speed by 1.5% every second"
  Drag factor: 0.015
- So:
  - After a second: $\vec{v} \leftarrow (1.0 - 0.015)\,\vec{v}$
  - After 2 seconds? $\vec{v} \leftarrow (1.0 - 0.015)^2\,\vec{v}$
  - After k seconds? $\vec{v} \leftarrow (1.0 - 0.015)^k\,\vec{v}$
  - After dt seconds? $\vec{v} \leftarrow (1.0 - 0.015)^{dt}\,\vec{v}$

    e.g. 1/60 = 0.17 sec
  - Which can be approximated with $\vec{v} \leftarrow (1.0 - 0.015 \cdot dt)\vec{v}$
  - The approximation is good when *this* is small

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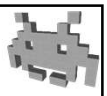
Equivalent formulas for friction
(controlled by different constants k_0 , k_1 , k_2)



- $\vec{f} = -k_0\vec{v}$ => $\vec{a} = -\frac{k_0}{m}\vec{v}$ => $\vec{v} \leftarrow \vec{v} - \frac{k_0}{m}\vec{v}\,dt$ =>

  update: $\vec{v} \leftarrow \left(1 - \frac{k_0}{m}\right)^{dt}\vec{v}$
- update: $\vec{v} \leftarrow (1 - k_1)^{dt}\vec{v}$
- update: $\vec{v} \leftarrow e^{-k_2 \cdot dt}\vec{v}$

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Equivalent formulas for friction
(controlled by different constants k_0 , k_1 , k_2) - notes

- 1st way: k_0 is the ratio force magnitude / speed.
  Physically based, and auto-adapts with mass,
  but difficult to tune in your game?
- 2nd way: k_1 (i.e., $\frac{k_0}{m}$) is the velocity loss per second (%).
  Intuitive to set! (here, we will adopt this way).
- 3rd way: k_2 (i.e., $-\log(1 - k_1)$)
  is the exponential decay.
  Also quite used in videogames
  (e.g., in Unity, Godot, Unreal)

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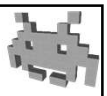
Velocity Damping: pseudo-code



```
Vec3 position = …
Vec3 velocity = …

void initState(){
   position = …
   velocity = …
}

void physicStep( float dt )
{
   Vec3 acceleration = force( positions ) / mass;
   position += velocity * dt;
   velocity += acceleration * dt;
   velocity *= (1.0 – DRAG * dt);
}

void main(){
  initState();
  while (1) do physicStep( 1.0 / FPS );
}
```

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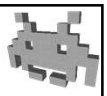
Velocity Damping: notes



- Velocity Damping is useful for robustness
  - Prevents the energy to ever increase
- Limitations:
  - it may exaggerate frictions of, e.g., air, especially in absence of contacts
  - it's a crude approximation: attrition forces are not really *linear* with speed
- In practice:
  - low drag: hardly noticeable (in the short run), increases robustness
  - high drag (e.g. 2% per sec): everything feels like to be moving in molasses (ita: *melassa*); everything quickly grinds to a halt
  - super high drag: (e.g. >25% per sec) basically, no inertia anymore. May be useful to converge to (local) minimal energy states: your simulator is now basically a solver for statics, not dynamics

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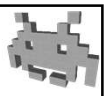
Continuity of pos and vel



- In real Newtonian physics the state (pos and vel) can only change *continuously*
  - No sudden jump!
- In practice, sometimes is useful to artificially break continuity in the simulations
- Discontinuous changes:
  - for positions: "teleports"
  - for velocity: "impulses"
  - In the real world, those variations can well be consequences of forces, but these forces are not modelled as such, in our system

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Dynamics displacements VS kinematic

a **discontinuous** change of state (position)

$$\ldots$$
$$\mathrm{p} = \mathrm{p} + \vec{\mathrm{v}} \cdot dt$$
$$\ldots$$

aka dynamic displacements

Justified by physics

$$\ldots$$
$$\mathrm{p} = \mathrm{p} + d\mathrm{p}$$
$$\ldots$$

aka Kinematic displacements

Just "teleportation"

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Impulses VS Forces

a **discontinuous** change of state (velocity)

…

$$\vec{v} = \vec{v} + \left(\vec{f} / m\right) \cdot dt$$

…

- Forces (continuous)
  - Continuous application
  - every frame

…

$$\vec{v} = \vec{v} + \left(\vec{i} / m\right)$$

…

- Impulses
  - Infinitesimal time
  - una tantum

they model very intense but short forces (such as impacts)

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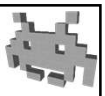
Impulses VS Forces



- Force :
  - it determines an acceleration
  - acc determines a (continuous!) change of vel
  - it's sustained over meaningful periods of time
- Impulse :
  - a (discontinuous!) change of vel
  - in reality: just a force with
    - very large magnitude
    - very short application times
  - we model it as a force "pre-multiplied" with its application time
  - it's useful to:
    - model phenomena with a time scale << *dt*
      e.g. a tennis ball rebounding against a tennis racket
    - control the simulation (direct change of velocity)

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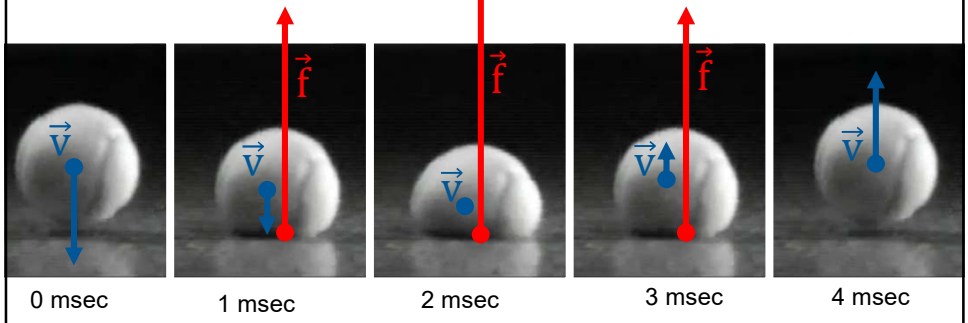
Impulses VS Forces

- what does *truly* happen when it bounces off the ground?



- very strong forces (but not infinite)
- applied for a very short time (but not instantaneous)
- see *collision response* later for details about the impulse based approximations

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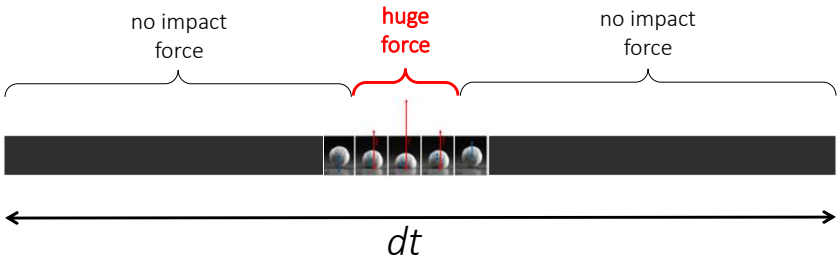
Impulses VS Forces

- what does *truly* happen when it bounces off the ground?



- This can only be modelled as an *impulse*, not a force
- *See also collision response, later*

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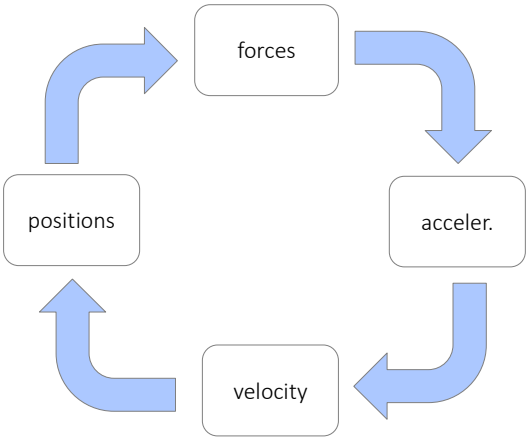
Next: better integration methods for (Newtonian) dynamics



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Other numerical integrators ("numerical ways to compute integrals")

- Some commonly used alternatives (among MANY!):
  - "Forward" Euler method (the one seen so far)
  - Symplectic Euler method
  - Leapfrog method
  - Verlet method
- These are just variants of each other – let's see them!
  - From the code point of view, no big change
  - They can differ in accuracy / behavior
  - They can have different "orders of accuracy"
  - Note: a more accurate method is also more efficient (larger dt are possible, so fewer steps are necessary)

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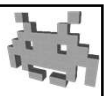
Forward Euler Method: limitations



- efficiency / accuracy: not too good
  - error accumulated over time = linear in dt
  - it's only a "first order" method
  - Doubles the steps = halve the *dt* , only halves the errors (can be better, but no guarantees)
- scarce stability for large *dt*
- minor problem: no reversibility, *even in theory*
  - real Newtonian Physics is reversible: flip all velocities and forces ⇒ go backward in time.
  - In our simulation (with Euler): this doesn't work exactly
  - Ability to go reverse a simulation would be useful in games! E.g. replays in a soccer game ?
  - Pro tip: basically, reverse time direction never done like this To go backward in time accurately, store states

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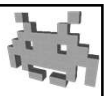
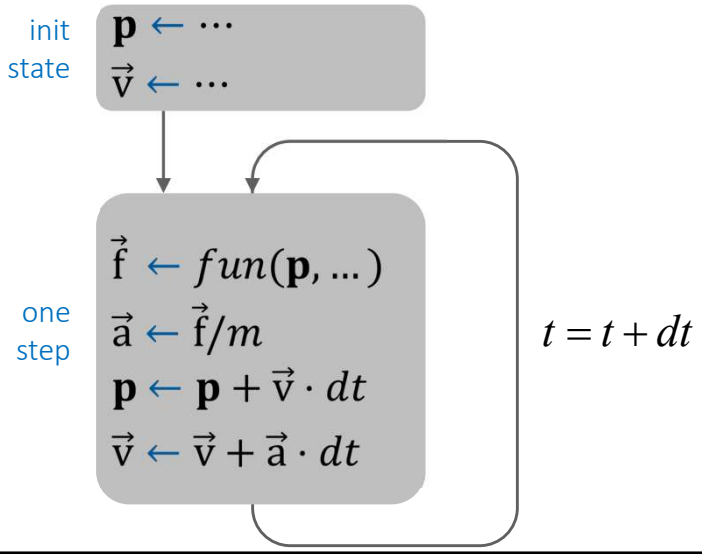
Forward Euler





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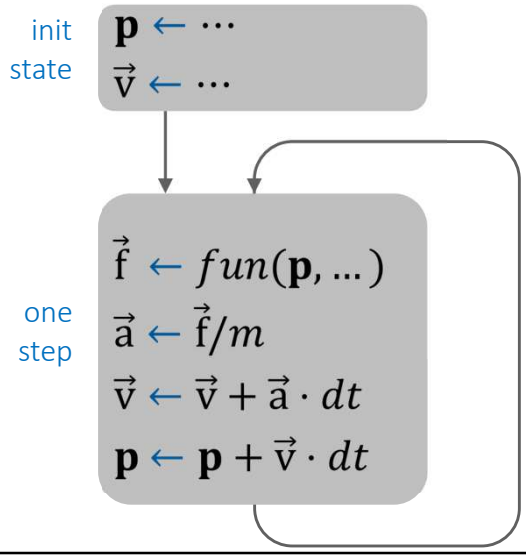
Symplectic Euler



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Forward Euler *pseudo code*

```
Vec3 position = …
Vec3 velocity = …

void initState(){
   position = …
   velocity = …
}

void physicStep( float dt )
{
   Vec3 acceleration = compute_force( position ) / mass;
   position += velocity     * dt;
   velocity += acceleration * dt;
}

void main(){
  initState();
  while (1) do physicStep( 1.0 / FPS );
}
```

Equivalent to…

$$\vec{f}_i \leftarrow function(p_i, \ldots)$$
$$\vec{a}_i \leftarrow \vec{f}/m$$
$$\vec{v}_{i+1} \leftarrow \vec{v}_i + \vec{a}_i \cdot dt$$
$$p_{i+1} \leftarrow p_i + \vec{v}_i \cdot dt$$

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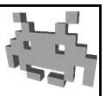
Symplectic Euler *pseudo code* (aka semi-implicit Euler)



Equivalent to...

$$\vec{f}_i \leftarrow function(p_i, \ldots)$$
$$\vec{a}_i \leftarrow \vec{f}/m$$
$$\vec{v}_{i+1} \leftarrow \vec{v}_i + \vec{a}_i \cdot dt$$
$$p_{i+1} \leftarrow p_i + \vec{v}_{i+1} \cdot dt$$

```
Vec3 position = …
Vec3 velocity = …

void initState(){
   position = …
   velocity = …
}

void physicStep( float dt )
{
   Vec3 acceleration = compute_force( position ) / mass;
   velocity += acceleration * dt;
   position +=     velocity * dt;
}

void main(){
  initState();
  while (1) do physicStep( 1.0 / FPS );
}
```

just flip the order

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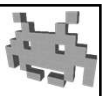
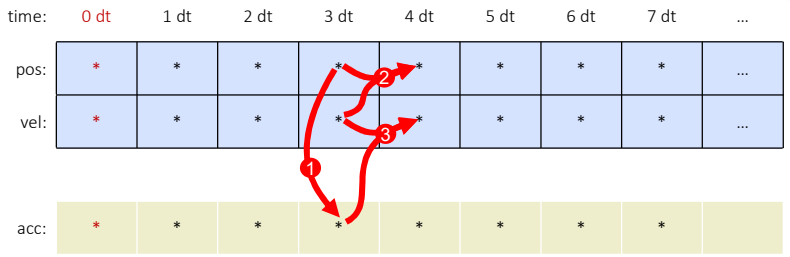
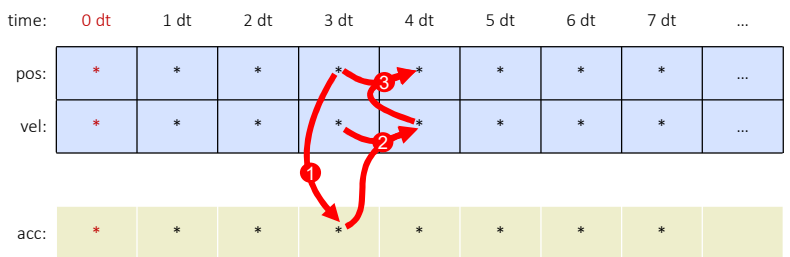
Forward Euler:





Symplectic Euler:



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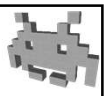
Forward Euler VS Symplectic Euler (warning: over-simplifications)



- From the code point of view, they are very similar
- The semantics changes:
  - in Symplectic Euler
    the position altered using *next frame* velocity
  - (it's "wrong", in a sense – but tends to work better)
- Similar properties, but better in practice
  - Same order of convergence (still just 1 ☹)
  - On average,
    Symplectic tends to be more stable and accurate

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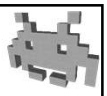
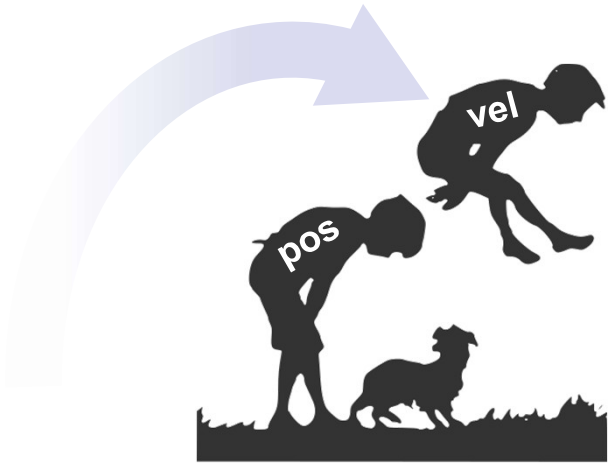
Leapfrog Integration Method





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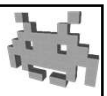
Leapfrog Integration Method



- Basic idea:
  store positions at time $k \cdot dt$
  (that is, at $t = 0, 1dt, 2dt, 3dt$...)
  but store velocities at time $k \cdot dt + ½\, dt$
  (that is, at $t = 0.5\, dt, 1.5\, dt, 2.5dt, 3.5dt$...)

- (roughly) equivalent to use a summatory of
  the areas of trapezoids,
  (having base dt and average height $v(t + ½\, dt)$)
  not rectangles, to compute the integral

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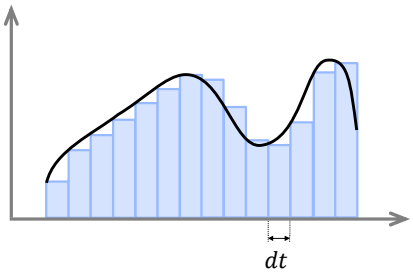
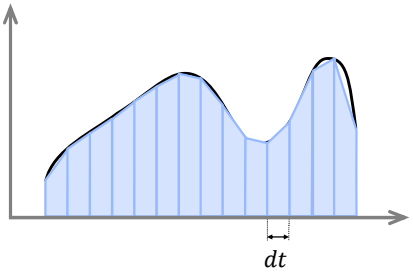
Better numerical integration (concept)



with Euler



with Leapfrog

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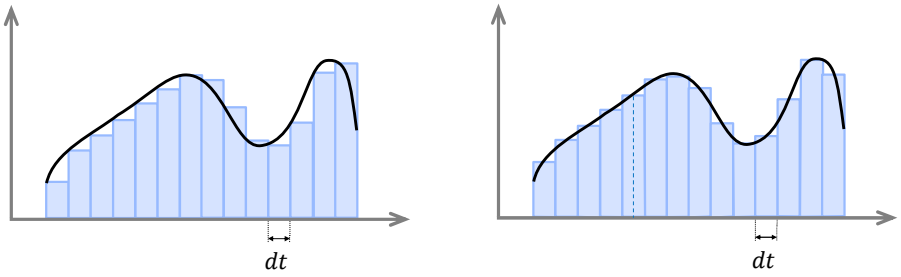
Better numerical integration (concept)



with Euler

with Leapfrog

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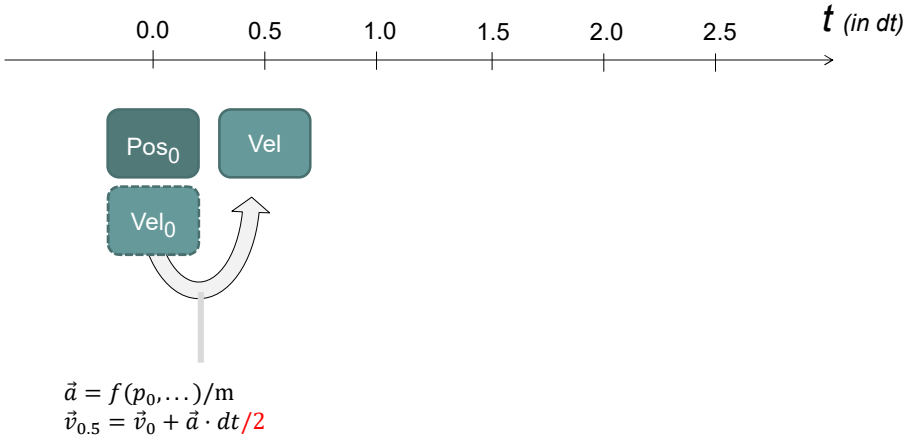
Leapfrog Integration first step



$\vec{a} = f(p_0, \ldots)/\mathrm{m}$

$\vec{v}_{0.5} = \vec{v}_0 + \vec{a} \cdot dt/2$

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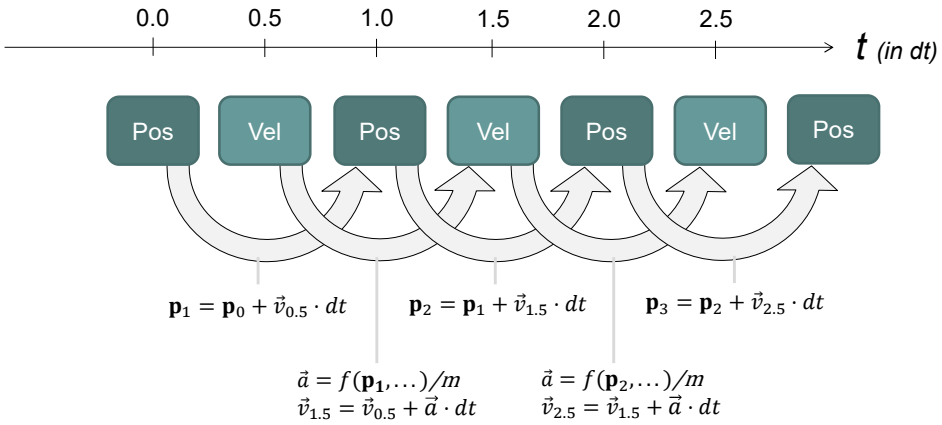
Leapfrog Integration



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Leapfrog method: pros and cons

- Same cost as Euler – and basically same code
  - Velocity stored in status = velocity "half a *dt* ago" (and after updating it: "half a frame in the future")
  - Only real difference: the initialization of velocities
- Better theorical accuracy, for the same *dt*
  - better asymptotic behavior: it's a "second order" system instead of first!
  - cumulated error: proportional to dt^2 instead of dt
  - error per frame: proportional to dt^3 instead of dt^2
- Bonus: **fully reversible**!
  - in theory only. Beware of numerical errors.
- But: **requires fixed *dt*** during all the simulation
  - Otherwise, updates of vel required in all particles

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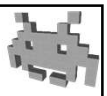
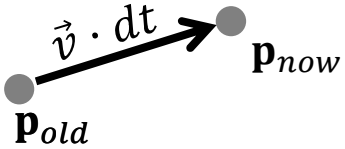
Verlet integration method



- Idea: remove velocity from state
  Instead, store previous position
- Velocity is now implicit
- It's defined by:
  - current pos $\mathbf{p}_{now}$
  - last pos $\mathbf{p}_{old}$
    which we need to record



$$\mathbf{p}_{now} = \mathbf{p}_{old} + \vec{v} \cdot dt$$ ← Euler & variants

$$\Longrightarrow$$

$$\vec{v} = (\mathbf{p}_{now} - \mathbf{p}_{old})/dt$$ ← Verlet

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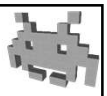
Verlet integration method: (modifying Euler integration…)



init state

$$\mathbf{p}_{now} = \dots$$
$$\mathbf{p}_{old} = \dots$$

one step

$$\vec{f} = funct(\mathbf{p}_{now}, \dots)$$
$$\vec{a} = \vec{f}/m$$
$$\vec{v} = (\mathbf{p}_{now} - \mathbf{p}_{old})/dt$$
$$\vec{v} = \vec{v} + \vec{a} \cdot dt$$
$$\mathbf{p}_{\text{next}} = \mathbf{p}_{now} + \vec{v} \cdot dt$$

expanding this…

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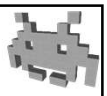
Verlet integration method (modifying Euler integration…)



init state

$\mathbf{p}_{now} \leftarrow \cdots$
$\mathbf{p}_{old} \leftarrow \cdots$

one step

$\vec{f} \leftarrow funct(\mathbf{p}_{now})$
$\vec{a} \leftarrow \vec{f}/m$
$\mathbf{p}_{next} \leftarrow 2\mathbf{p}_{now} - \mathbf{p}_{old} + \vec{a} \cdot dt^2$

$\mathbf{p}_{old} \leftarrow \mathbf{p}_{now}$
$\mathbf{p}_{now} \leftarrow \mathbf{p}_{next}$

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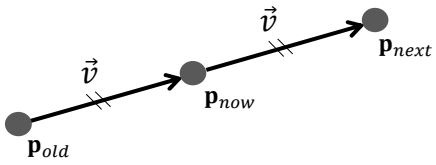
Verlet integration method: geometric interpretation



$\mathbf{p}_{next} = 2 \cdot \mathbf{p}_{now} - 1 \cdot \mathbf{p}_{old}$

$\mathbf{p}_{next}$ can be written as an extrapolation of $\mathbf{p}_{now}$, $\mathbf{p}_{old}$:

$\mathbf{p}_{next} = mix(\mathbf{p}_{old}, \mathbf{p}_{now}, 2)$

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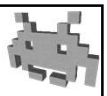
Verlet: characteristics



- Velocity is kept implicit
  - but that doesn't save RAM:
    we need to store previous position instead
  - (a point instead of a vector: same memory)
- Good efficiency / accuracy ratio
  - accumulated error: order of dt^2 (second order method)
- Extra bonus: reversibility
  - it's possible to go backward* in *t* and
    reach the initial state from any state
    * (just swap p_now and p_old)
  - only in theory… careful with implementation details

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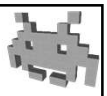
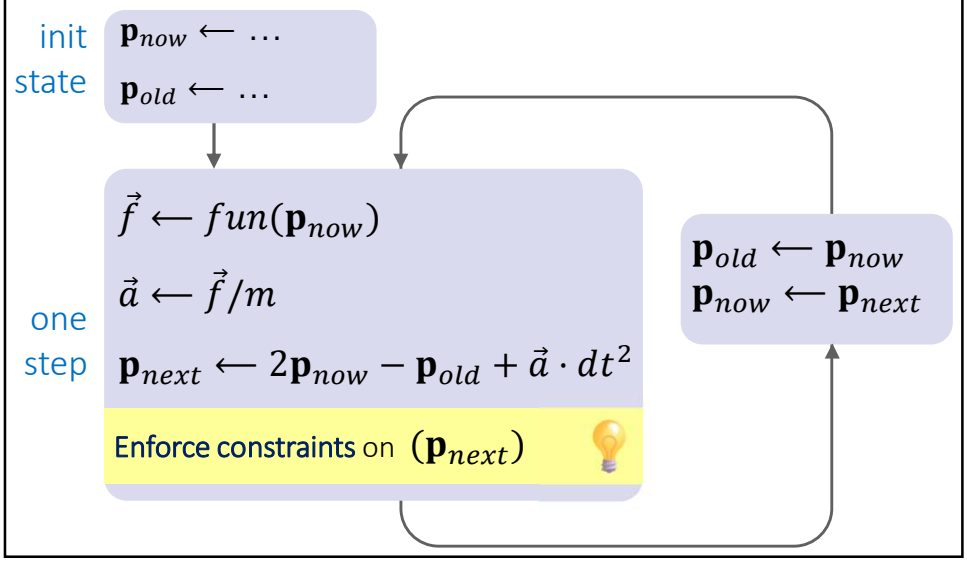
Verlet integration + "Position Based Dynamics" (PBD)





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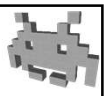
Position Based Dynamics (PDB)



- A positional constraint is an equality/inequality (a formula with '=' '>' '<' etc.) involving the *positions* of particles.
  - Useful, for example, to model consistency conditions
  - Like "*solid objects don't compenetrate each other*", or "*steel bars won't become shorter or longer than they are*"
  - We will see many examples
- We enforce (impose) positional constraint directly by displacing the *positions* of particles
  - Thanks to Verlet: this displacement automatically causes some appropriate update of the velocity!
  - it's not necessarily correct, but it's plausible and robust

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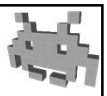
Example of a positional constraint



«I want all particles to stay above ground (that is, their y must never be negative) »

Enforce this constraint: trivial!

```
for (each particle i)
{
  if (p[i].y < 0) p[i].y = 0;
}
```

Imposing constraints like this one is a first part of collision response. For re-bounces, impulses must still be added (see collisions).

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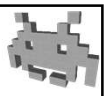
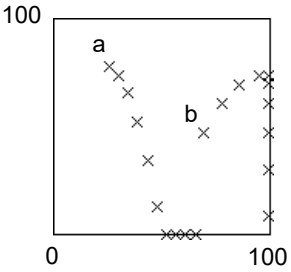
Example of a positional constraint (here, in 2D physics)



«I want particles to stay inside a 2D box [0 – 100] x [0 – 100] *»*



Enforce this constraint: simple clamp!

```
for (each particle i)
{
  p[i].x = clamp( p[i].x, 0, 100 );
  p[i].y = clamp( p[i].y, 0, 100 );
}
```

Imposing constraints like this one is a first part of collision response. For re-bounces, impulses must still be added (see collisions).

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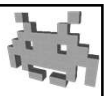
Verlet + Position Based Dynamics. Advantages



- flexibility: different constraints can be used to model many different phenomena
  - Useful constraints are straightforward to define
  - They are easy to impose (they involve only few particles)
  - They can be used to model many possible phenomena
  - See following slides for examples
- robustness : plausibility is ensured by *explicitly* enforcing the conditions we want to see
  - For example: a ball won't ever be seen outside the box containing it – and it will also recover from mistakes
- No forces / impulses are needed to enforce any such consistency conditions

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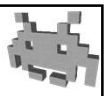
Verlet: *caveats*
(see next slides for solutions)



- ⚠ it assumes a constant *dt* (time-step duration)
  - if *dt* varies: corrections are needed! (how?)
- ⚠ Q: how to act on velocity (which is now implicit)?
  - for example, how to apply impulses ?
  - A: change $\mathbf{p}_{old}$ instead (how?)
- ⚠ Q: how to act of positions w/o impacting velocity?
  - for example, to apply teleports / kinematic motions ?
  - A: change both $\mathbf{p}_{new}$ and $\mathbf{p}_{old}$ (how?)
- ⚠ Q: how to apply velocity damps?

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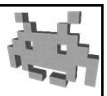
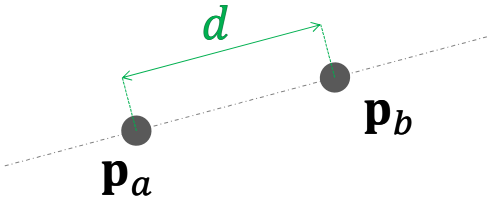
Example of positional constraint: equidistance constraint



*«Particles **a** and **b** must stay at a fixed distance **d** »*



I require that... $\|\mathbf{p}_a - \mathbf{p}_b\| = d$

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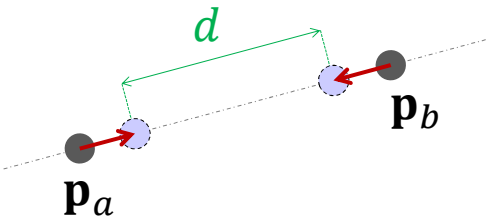
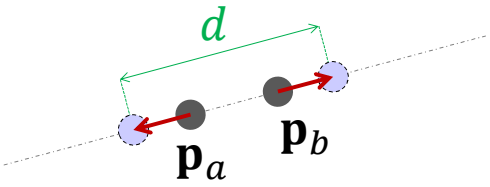
Enforce equidistance constraints
(assuming equal masses for now)

if $\|\mathbf{p}_a - \mathbf{p}_b\| > d$



if $\|\mathbf{p}_a - \mathbf{p}_b\| < d$



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Enforce equidistance constraints: pseudo code

```
Vector3 pa, pb; // curr positions of a,b
float d;        // distance (to enforce)

Vector3 v = pa - pb;
float currDist = v.length;

v /= currDist;  // normalization of v

float delta = currDist - d ;

pa += ( 0.5 * delta) * v;
pb -= ( 0.5 * delta) * v;
```

we move each particle *half the way*
(it makes sense, but see later for why exactly)

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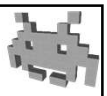
Compare: equidistance constraints vs. springs



- Similar
  - they both mean: these 2 particles "want to be" at *this* distance (not more, not less)

some constant scalar parameter D

- but different
  - equidistance constraint:
    - applied during constraint enforcement
    - directly affects positions
    - models a rigid rod between the two particles
      - of a given length
    - sometimes called a "HARD" constraint
  - spring:
    - applied during force evaluation step
    - affects forces, therefore accelerations
    - models a deformable spring between the two particles
      - of a given length
    - sometimes called a "SOFT" constraint
- A physic engine can use both at the same time!

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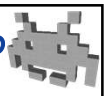
How to enforce a positional constraiont? (see next lecture for the complete anwser)



When enforcing constraints…

- If a constraint is valid, no problem.
- If it's not, change particle positions so that it does.
  - Problems: there's usually many possible ways to do

⚠ Which one to pick?

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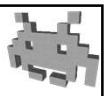
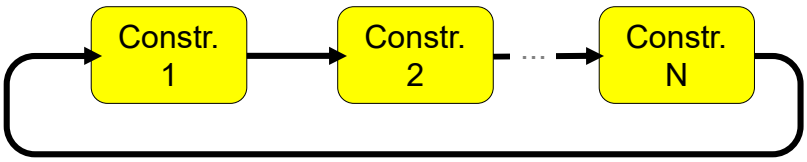
Enforcing a set of constraints



- There are many constraints to impose:
  when you solve one → maybe you break another!
- Simultaneous enforcement: computationally expensive

- Practical & easy solution: just enforce them in cascade
  (similar in concept to Gauss-Seidel solvers):



Repeat until convergence (= max error below threshold)
...but at most for N_{MAX} times! (remember: our simulation is *soft* real-time)

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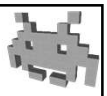
Enforcing a set of constraints one after the other (in cascade)



- The whole loop for imposing the constraints happen in the constraint enforcement phase on one physics step
- Notes about convergence:
  - needed iterations (typically) few: e.g. 1 ~ 10 (efficient!).
  - if convergence not reached within a given number of steps: never mind, next frames will fix it (so it's robust)
  - (it is never reached, if constraints are contradictory)
  - Optimization (to reduce the number of needed iterations): solve the most unsatisfied constraints first

⚠ Problem: it's a sequential approach! ☹
  - parallelized versions (similar to Jacobi solvers) are possible
  - they have a worse convergence in practice
    (they require more iterations), but each iteration is faster

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Compounds of particles disguised as rigid bodies

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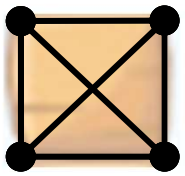
We can combine equidistance constraints to obtain rigid objects!

- Rigid body dynamics as emerging behavior
  - without explicitly keeping track their orientation, angular vel, angular acc., etc.



A box in 2D?
(rigid object)
A configuration of:
- 4 particles
- 6 equidistance constraints

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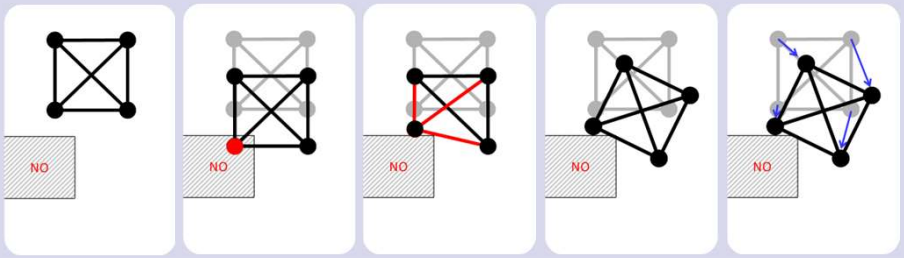
Example



FRAME 0

FRAME 1
dynamics

FRAME 1
constraints solving
(repeat until
convergence)

FRAME 1
final
-
implicit velocities
shown in blue

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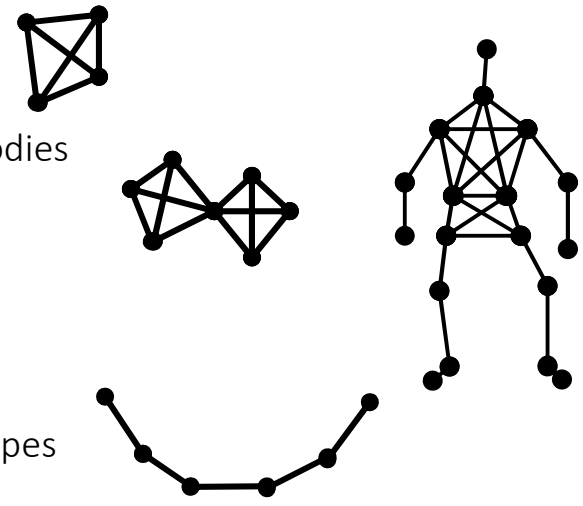
We can combine equidistance constraints to obtain…

- Rigid bodies
- Articulated bodies
- Ragdolls
- Cloth
- Non-elastic ropes
- …and more



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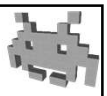
Positional constraints (in general terms, and more formally)



- A predicate defined on the position(s) of a number of particles (usually, a small number: 1 - 4)

$$\mathcal{C}: (\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \dots) \rightarrow \{ true, false \}$$

- For example, the equidistance constraints is

$$\mathcal{C}(\mathbf{p}_a, \mathbf{p}_b) \Leftrightarrow \|\mathbf{p}_a - \mathbf{p}_b\| = k_{CONST}$$

- They can be an equality ($=$) or an inequality ($\leq$ or $\geq$)

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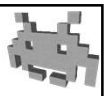
Equality positional constraints: examples



- Equidistance constraint (the one we have seen):
  «these 2 particles must stay at distance k»
  - E.g: because they are linked by a metal rod of length k
- Fixed positions:
  «this particle must stay in position $\mathbf{p}_a$ »
  - the particle is “pinned” in position
  - trivial to impose, but still useful!
- Coplanarity / collinearity:
  «these N particles must stay on a line / on a plane»

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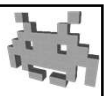
Equality positional constraints: other examples



- Volume preservation:
  "The volume delimited by the squishy ballon defined by these particles is a constant k_{CONST} *"*
  (e.g., because it's filled with water)

- How to impose it (approximation):
  1. Estimate current total volume v
  2. uniform scale the entire object by factor $\sqrt[3]{f_{\text{CONS}} \ / v}$ around its barycenter

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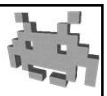
Inequality positional constraints: example



- "please don't sink below the ground"
  assuming the ground is the plane Y = 0

$$C(\mathbf{p}_a) \Leftrightarrow \mathbf{p}_a.y \geq 0$$

- Trivial to impose:
  just set the y to 0, if it is < 0

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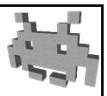
Inequality positional constraints: example



- "this particle must stay above this fixed (and arbitrary) plane"
  - For example, because the plane is a solid unmovable slab
  - The plane is given by a point on it $\mathbf{p}_\mathrm{q}$ and its normal $\hat{n}_q$

$$\mathrm{C}(\mathbf{p}_\mathrm{a}) \Leftrightarrow (\mathbf{p}_\mathrm{a} - \mathbf{p}_\mathrm{q}) \cdot \hat{n}_q \geq 0$$

- To enforce it (when it's not already true) enforce

$$(\mathbf{p}_\mathrm{a} - \mathbf{p}_\mathrm{q}) \cdot \hat{n}_q = 0$$

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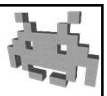
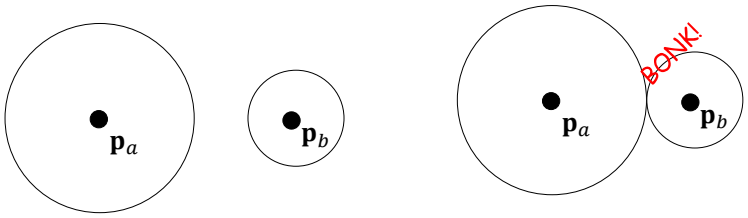
Inequality positional constraints: example



- These two particles must be *at least* k_{CONST} apart

$$\mathcal{C}(\mathbf{p}_\mathrm{a}, \mathbf{p}_\mathrm{b}) \Leftrightarrow \|\mathbf{p}_\mathrm{a} - \mathbf{p}_\mathrm{b}\| \geq k_{CONST}$$

  - For example, because they are the centers of two rigid spheres and k_{CONST} is the sum of their radii
  - part of "collision handling" (a topic for later)



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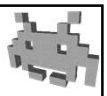
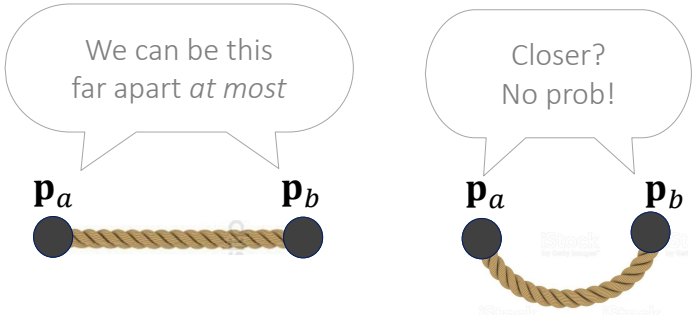
Inequality positional constraints: example



- These two particles must be *at most* k_{CONST} apart

$$\mathcal{C}(\mathbf{p}_a, \mathbf{p}_b) \Leftrightarrow \|\mathbf{p}_a - \mathbf{p}_b\| \leq k_{CONST}$$

  - For example, because they are tied by an inextensible rope that has length k_{CONST} (but can fold)



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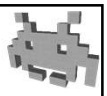
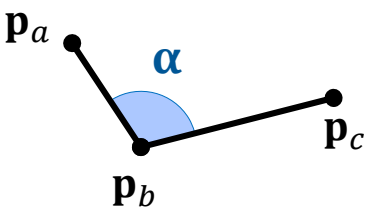
Inequality positional constraints: example



- Angle constraints, e.g. $\boldsymbol{\alpha} < \boldsymbol{\alpha}_{\max}$ with $\boldsymbol{\alpha}$ the angle between $\mathbf{p}_a, \mathbf{p}_b$ and $\mathbf{p}_b, \mathbf{p}_c$
  - e.g., on joints: *«elbows cannot bend backward»*
  - (a constraint between three particles!)



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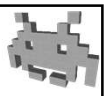
Enforcing one positional constraint (in general terms)



- Inequality constraint:
  1. *Test:* does the inequality already hold?
  2. If so: do nothing
  3. If not: enforce it as an equality (=) instead (see below)
- Equality constraint:
  - All involved particles must be displaced from that current position, so that it now holds
  - There can be infinite ways to achieve this!
    Question: Which one to pick?

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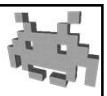
Enforcing one equality constraint: (assuming for now all particles have same mass)



- Answer:
  minimize the sum of *squared* displacements
  (with respect to current position)
- For each kind of constraint, we need to find the minimizer analytically
  ("analytically" = in closed form = "with formulas" = "solving a simple math problem on paper")
  - That's what we did for the equality constraint

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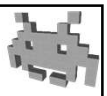
Enforcing one equality constraint
(assuming for now all particles have same mass)



- We want to enforce a constraint $\mathcal{C}$ on particles $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$ currently in positions $\mathbf{p}_\mathrm{a}, \mathbf{p}_\mathrm{b}, \mathbf{p}_\mathrm{c}$ …

$$\mathcal{C}: (\mathbf{p}_\mathrm{a}, \mathbf{p}_\mathrm{b}, \mathbf{p}_\mathrm{c}, \ldots) \rightarrow \{ true, false \}$$

- We must apply the displacements $\overrightarrow{d_\mathrm{a}}$, $\overrightarrow{d_\mathrm{b}}$, $\overrightarrow{d_\mathrm{c}}$ that are the

$$\underset{\overrightarrow{d_\mathrm{a}}, \overrightarrow{d_\mathrm{b}}, \overrightarrow{d_\mathrm{c}}, \ldots}{\operatorname{argmin}} \left(\|\overrightarrow{d_\mathrm{a}}\|^2 + \|\overrightarrow{d_\mathrm{b}}\|^2 + \|\overrightarrow{d_\mathrm{c}}\|^2 + \cdots \right)$$

$$\text{such that } \mathcal{C}\left(\mathbf{p}_\mathrm{a} + \overrightarrow{d_\mathrm{a}}, \mathbf{p}_\mathrm{b} + \overrightarrow{d_\mathrm{b}}, \mathbf{p}_\mathrm{c} + \overrightarrow{d_\mathrm{c}}, \ldots\right)$$

among all the choices that satisfy this,
we want the one which minimizes this

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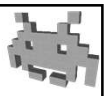
Enforcing one equality constraint
(in general, for particles with different mass)



- We want to enforce a constraint $\mathcal{C}$ on particles $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$ in positions $\mathbf{p}_\mathrm{a}, \mathbf{p}_\mathrm{b}, \mathbf{p}_\mathrm{c}$ … and with masses $m_\mathrm{a}, m_\mathrm{b}, m_\mathrm{c}, \ldots$

$$\mathcal{C}: (\mathbf{p}_\mathrm{a}, \mathbf{p}_\mathrm{b}, \mathbf{p}_\mathrm{c}, \ldots) \rightarrow \{ true, false \}$$

- We must apply the displacements $\overrightarrow{d_\mathrm{a}}$, $\overrightarrow{d_\mathrm{b}}$, $\overrightarrow{d_\mathrm{c}}$ which are the:

$$\underset{\overrightarrow{d_\mathrm{a}}, \overrightarrow{d_\mathrm{b}}, \overrightarrow{d_\mathrm{c}}, \ldots}{\operatorname{argmin}} \left(m_\mathrm{a}\|\overrightarrow{d_\mathrm{a}}\|^2 + m_\mathrm{b}\|\overrightarrow{d_\mathrm{b}}\|^2 + m_\mathrm{c}\|\overrightarrow{d_\mathrm{c}}\|^2 + \cdots \right)$$

$$\text{such that } \mathcal{C}\left(\mathbf{p}_\mathrm{a} + \overrightarrow{d_\mathrm{a}}, \mathbf{p}_\mathrm{b} + \overrightarrow{d_\mathrm{b}}, \mathbf{p}_\mathrm{c} + \overrightarrow{d_\mathrm{c}}, \ldots\right)$$

among all the choices that satisfy this,
we want the one which minimizes this

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